Division by Zero Calculus (Draft)

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Abstract: The common sense on the division by zero with a long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on derivatives we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In this book, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$. In particular, we will introduce several fundamental concepts on calculus, Euclidian geometry, analytic geometry, complex analysis and differential equations. We will see new properties on the Laurent expansion, singularity, derivative, extension of solutions of differential equations beyond analytical and isolated singularities, and reduction problems of differential equations. On Euclidean geometry and analytic geometry, we will find new fields by the concept of the division by zero. We will collect many concrete properties in the mathematical sciences from the viewpoint of the division by zero. We will know that the division by zero is our elementary and fundamental mathematics.

Key Words: Division by zero calculus, singularity, derivative, differential equation, division by zero, $0/0 = 1/0 = z/0 = 0$, $\tan(\pi/2) = 0$, $\log 0 = 0$, infinity, discontinuous, point at infinity, gradient, Laurent expansion, extension of solutions of differential equations, reduction problems of

differential equations, analytic geometry, singular integral, conformal mapping, Euclidean geometry, Wazan.

Preface

The division by zero has a long and mysterious story over the world (see, for example, H. G. Romig [45] and Google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628. In particular, note that Brahmagupta (598 -668 ?) established the four arithmetic operations by introducing 0 and at the same time he defined as $0/0 = 0$ in Brhmasphuasiddhnta. Our world history, however, stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we will see that his definition is right and suitable.

The division by zero $1/0 = 0/0 = z/0$ itself will be quite clear and trivial with several natural extensions of the fractions against the mysterously long history, as we can see from the concepts of the Moore-Penrose generalized inverses or the Tikhonov regularization method to the fundamental equation $az = b$, whose solution leads to the definition $z = b/a$.

However, the result (definition) will show that for the elementary mapping

$$W = \frac{1}{z}, \tag{0.1}$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [60, 61, 62]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([27, 35]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidean geometry, analytic geometry, differential equations, complex analysis in the undergraduate level and to our basic ideas for the space and universe.

We have to arrange globally our modern mathematics in our undergraduate level. Our common sense on the division by zero will be wrong, with our basic idea on the space and the universe since Aristotle and Euclid. We would like to show clearly these facts in this book. The content is in the undergraduate level.

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Meanwhile, we are having interesting negative comments from several people on our division by zero. However, they seem to be just traditional and old feelings, and they are not reasonable at all for the author. The typical good comment for the first draft is given by some physician as follows:

Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out (2017.10.14.08:55).

The detailed research procedures and many ideas on the division by zero are presented in the Announcements as in stated in the references and in Japanese Announcements:

148(2014.2.12), 161(2014.5.30), 163(2014.6.17), 188(2014.12.15), 190(2014.12.24), 191(2014.12.27), 192(2014.12.27),

194(2015.1.2), 195(2015.1.3), 196(2015.1.4), 199(2015.1.15), 200(2015.1.16), 202(2.015.2.2), 215(2015.3.11), 222(2015.4.8), 225(2015.4.23), 232(2015.5.26), 249(2015.10.20), 251(2015.10.27), 253(2015.10.28,255(2015.11.3), 257(2015.11.05), 259(2015.12.04), 262(2015.12.09),

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Contents

1 Introduction

By a **natural extension** of the fractions

$$\frac{b}{a} \tag{1.1}$$

for any complex numbers a and b, we found the simple and beautiful result, for any complex number b

$$\frac{b}{0} = 0, \tag{1.2}$$

incidentally in [49] by the Tikhonov regularization for the Hadamard product inversions for matrices and we discussed their properties and gave several physical interpretations on the general fractions in [22] for the case of real numbers. The result is a very special case for general fractional functions in [13].

The division by zero has a long and mysterious story over the world (see, for example, H. G. Romig [45] and Google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628. In particular, note that Brahmagupta (598 -668 ?) established the four arithmetic operations by introducing 0 and at the same time he defined as $0/0 = 0$ in Brhmasphuasiddhnta. Our world history, however, stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we will see that his definition is right and suitable.

Indeed, we will show typical examples for $0/0 = 0$. However, in this introduction, these examples are based on some natural feelings and are not mathematics, because we do still not give the definition of $0/0$. However, following our new mathematics, these examples may be accepted as natural ones later:

The conditional probability $P(A|B)$ for the probability of A under the condition that B happens is given by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, then, of course, $P(A \cap B) = 0$ and from the meaning, $P(A|B) = 0$ and so, $0/0 = 0$.

For the representation of inner product in vectors

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{AB}$$

$$= \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}},$$

if **A** or **B** is the zero vector, then we see that $0 = 0/0$. In general, the zero vector is orthogonal for any vector and then, $\cos\theta = 0$.

For this paragraph for our old version, Professor J. Czajko gave kindly the detailed comments following his some general idea for the division by zero under the multispatial reality paradigm and stated in the last part:

As one can see, the single-space reality paradigm, which is unspoken of in the former mathematics and physics, creates tacitly evaded inconsistencies even at the logical level of mathematical reasonings.

Dieudonne ([15]) has also tentatively assumed $xy = 0$ wherever one of the variables is 0 and the other ∞ [*], which is similar to $0/0 = 0$. Besides, if your formula (1.2) would be rendered as $b/0 = 0 + i0$ then it might lead one to question whether or not the still reigning single-space reality paradigm is admissible in general.

[*] Dieudonn J. Treatise on analysis II. NewYork: Academic Press, 1970, p.151.

Look his basic great references, [10, 11].

For the differential equation

$$\frac{dy}{dx} = \frac{2y}{x},$$

we have the general solution with constant C

$$y = Cx^2.$$

At the origin $(0,0)$ we have

$$y'(0) = \frac{0}{0} = 0.$$

For three points a, b, c on a circle with center at the origin on the complex z-plane with radius R, we have

$$|a + b + c| = \frac{|ab + bc + ca|}{R}.$$

If $R = 0$, then $a, b, c = 0$ and we have $0 = 0/0$.

For a circle with radius R and for an inscribed triangle with side lengths a, b, c, and further for the inscribed circle with radius r for the triangle, the area S of the triangle is given by

$$S = \frac{r}{2}(a + b + c) = \frac{abc}{4R}. \tag{1.3}$$

If $R = 0$, then we have

$$S = 0 = \frac{0}{0} \tag{1.4}$$

(H. Michiwaki: 2017.7.28).

For the second curvature

$$K_2 = \left((x'')^2 + (y'')^2 + (z'')^2\right)^{-1} \cdot \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}, \tag{1.5}$$

if $(x'')^2 + (y'')^2 + (z'')^2 = 0$; that is, when for the case of lines, then $0 = 0/0$.

For the function sign $x = x/|x|$, we have, automatically, sign $x = 0$ at $x = 0$.

We have furthermore many and concrete examples as we will see in this book.

However, we do not know the reason and motivation of the definition of $0/0 = 0$ by Brahmagupta, furthermore, for the important case $1/0$ we do not know any result there. – Indeed, we find many and many wrong logics on the division by zero, without the definition of the division by zero $z/0$. However, Sin-Ei Takahasi ([22]) discovered a simple and decisive interpretation (1.2) by analyzing the extensions of fractions and by showing the complete characterization for the property (1.2):

Proposition 1. *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying*

$$F(b, a)F(c, d) = F(bc, ad)$$

for all

$$a, b, c, d \in \mathbf{C}$$

and

$$F(b, a) = \frac{b}{a}, \quad a, b \in \mathbf{C}, a \neq 0.$$

Then, we obtain, for any $b \in \mathbf{C}$

$$F(b, 0) = 0.$$

Note that the complete proof of this proposition is simply given by 2 or 3 lines, as we will give its complete proof later. In order to confirm the uniqueness result by Professor Takahasi, Professor Matteo Dalla Riva gave the proposition independently of Professor Takahasi as stated in ([22]). Indeed, when the Takahasi's result was informed, he was first negative for the Takahasi uniqueness theorem.

In a long mysterious history of the division by zero, this proposition seems to be decisive. Since the publication of the paper, over fully four years we see still curious information on the division by zero and we see still many wrong opinions on the division by zero.

Indeed, the Takahasi's assumption for the product property should be accepted for any generalization of fraction (division). Without the product property, we will not be able to consider any reasonable fraction (division).

Following the proposition, we should **define**

$$F(b, 0) = \frac{b}{0} = 0,$$

and consider, for any complex number b, as (1.2); that is, for the mapping

$$W = \frac{1}{z}, \tag{1.6}$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [60, 61, 62]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([27, 35]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidian geometry, analytic geometry, complex analysis and the theory of differential equations in an undergraduate level and furthermore to our basic ideas for the space and universe.

Meanwhile, the division by zero (1.2) was derived from several independent approaches as in:

1) by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized inverse to the fundamental equation $az = b$ that leads to the definition of the fraction $z = b/a$,

2) by the intuitive meaning of the fractions (division) by H. Michiwaki,

3) by the unique extension of the fractions by S. Takahasi, as in the above,

4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ into $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$,

and

5) by considering the values of functions with the mean values of functions.

Furthermore, in ([26]) we gave the results in order to show the reality of the division by zero in our world:

A) a field structure containing the division by zero — the **Yamada field Y**,

B) by the gradient of the y axis on the (x, y) plane — $\tan \frac{\pi}{2} = 0$,

C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with center at the origin on the complex z plane — the reflection point of zero is zero, (The classical result is wrong, see [35]),

and

D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

Furthermore, in ([27],[49]), we discussed many division by zero properties in the Euclidean plane - however, precisely, our new space is not the Euclidean space. More recently, we see the great impact to Euclidian geometry in connection with Wazan in ([36, 37, 38]). In ([23]), we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero.

See J. A. Bergstra, Y. Hirshfeld and J. V. Tucker [7] and J. A. Bergstra [8] for the relationship between fields and the division by zero, and the importance of the division by zero for computer science. It seems that the relationship of the division by zero and field structures are abstract in their papers.

Meanwhile, Carlstrm ([9]) introduced the wheel theory; wheels are a type of algebra where division is always defined. In particular, division by zero is meaningful. The real numbers can be extended to a wheel, as can any commutative ring. The Riemann sphere can also be extended to a wheel by adjoining an element $\perp$, where $0/0 = \perp$. The Riemann sphere is an extension of the complex plane by an element ∞, where $z/0 = \infty$ for any complex $z \neq 0$. However, $0/0$ is still undefined on the Riemann sphere, but is defined in its extension to a wheel. The term wheel is introduced by the topological picture $\odot$ of the projective line together with an extra point $\perp = 0/0$.

Similarly, T.S. Reis and J.A.D.W. Anderson ([43, 44]) extends the system of the real numbers by defining division by zero with three infinities $+\infty, -\infty.\Phi$ (Transreal Calculus).

However, we can introduce simply a very natural algebra containing the division by zero that is a natural extension (modification) of our mathematics, as the Yamada field.

In connection with the deep problem with physics of the division by zero problem, see J. Czajko [10, 11].

J. P. Barukcic and I. Barukcic ([6]) discussed the relation between the division 0/0 and special relative theory of Einstein. However it seems that their result is curious with their logics. Their results contradict with ours.

L.C. Paulson stated that I would guess that Isabelle has used this **convention** $1/0 = 0$ since the 1980s and introduced his book [32] referred to this fact. However, in his group the importance of this fact seems to be entirely ignored at this moment as we see from the book.

For the recent great works, see E. Jebek and B. Santangelo [20, 52]. From their abstracts,

For any sufficiently strong theory of arithmetic, the set of Diophantine equations provably unsolvable in the theory is algorithmically undecidable, as a consequence of the MRDP theorem. In contrast, we show decidability of Diophantine equations provably unsolvable in Robinson's arithmetic Q. The argument hinges on an analysis of a particular class of equations, hitherto unexplored in Diophantine literature. We also axiomatize the universal fragment of Q in the process.

and

The purpose of this paper is to emulate the process used in defining and learning about the algebraic structure known as a Field in order to create a new algebraic structure which contains numbers that can be used to define Division By Zero, just as i can be used to define $\sqrt{-1}$.

This method of Division By Zero is different from other previous attempts in that each $\frac{\alpha}{0}$ has a different unique, numerical solution for every possible α, albeit these numerical solutions are not any numbers we have ever seen. To do this, the reader will be introduced to an algebraic structure called an S-Structure and will become familiar with the operations of addition, subtraction, multiplication and division in particular S-Structures. We will build from the ground up in a manner similar to building a Field from the ground up. We first start with general S-Structures and build upon that to S-Rings and eventually S-Fields, just as one begins learning about Fields by first understanding Groups, then moving up to Rings and ultimately to Fields. At each step along the way, we shall prove important properties of each S-Structure and of the operations in each of these S-Structures. By the end, the reader will become familiar with an S-Field, an S-Structure which is an extension of a Field in which we may uniquely define $\alpha/0$ for every non-zero α which belongs to the Field. In fact, each $\frac{\alpha}{0}$ has a different, unique solution for every possible α. Furthermore, this Division By Zero satisfies $\alpha/0 = q$ such that $0 \cdot q = \alpha$, making it a true Division Operation,

respectively.

Meanwhile, we should refer to up-to-date information:

Riemann Hypothesis Addendum - Breakthrough Kurt Arbenz :

https://www.researchgate.net/publication/272022137 Riemann Hypothesis Addendum - Breakthrough.

Here, we recall Albert Einstein's words on mathematics: Blackholes are where God divided by zero. I don't believe in mathematics. George Gamow

(1904-1968) Russian-born American nuclear physicist and cosmologist remarked that "it is well known to students of high school algebra" that division by zero is not valid; and Einstein admitted it as **the biggest blunder of his life** (Gamow, G., My World Line (Viking, New York). p 44, 1970).

In this book, we will discuss the division by zero in calculus and Euclidian geometry and introduce various applications to differential equations and others, and we will be able to see that the division by zero is our elementary and fundamental mathematics.

In order to realize our long and wrong basic ideas for the point at infinity and the mirror image with respect to a circle, we refer to the properties of the stereographic projection and the mirror image in details in Sections 3 and 4.

This book is an extension of the sourse file ([42]) of the talk presented at the International Conference:

https://sites.google.com/site/sandrapinelas/icddea-2017

In this book, we would like to present clearly the conclusion of the talk:

The division by zero is uniquely and reasonably determined as

$$1/0 = 0/0 = z/0 = 0$$

in the natural extensions of fractions.

We have to change our basic ideas for our space and world.

We have to change globally our textbooks and scientific books on the division by zero.

2 Introduction and definitions of general fractions

We first introduce several definitions of our general fractions containing the division by zero.

2.1 By the Tikhonov regularization

For any real numbers a and b containing 0, we will introduce general fractions

$$\frac{b}{a}. \tag{2.1}$$

We will think that for the fraction (2.1), it will be given by the solution of the equation

$$ax = b.$$

Here, in order to see its essence, we will consider all on the real number field $\mathbf{R}$. However, for $b \neq 0$, this equation has not any solution for the case $a = 0$, and so, by the concept of the Tikhonov regularization method, we will consider the equation as follows:

For any fixed $\lambda > 0$, the minimum member of the Tikhonov function in x

$$\lambda x^2 + (ax - b)^2; \tag{2.2}$$

that is,

$$x_\lambda(a, b) = \frac{ab}{\lambda + a^2} \tag{2.3}$$

may be considered as the fraction in the sense of Tikhonov with parameter λ, in a generalized sense. By taking the limit

$$\lim_{\lambda \to +0} x_\lambda(a, b) = \frac{b}{a}, \tag{2.4}$$

we will define the general fractions.

Note that, for $a \neq 0$, the definition (2.4) is the same as the ordinary sense, however, when $a = 0$, we obtain the desired results $b/0 = 0$, since $x_\lambda(0, b) = 0$, always.

The result (2.4) is, of course, a trivial Moore-Penrose generalized inverse (solution) for the equation $ax = b$.

For the general theory of the Tikhonov regularization and many applications, see the cited references, for example, [50].

2.2 By the Takahasi uniqueness theorem

Sin-Ei, Takahashi ([54]) established a simple and natural interpretation (1.2) by analyzing any extensions of fractions and by showing the complete characterization for such property (1.2). Furthermore, he examined several fundamental properties of the general fractions. His result will show that our mathematics says that the results (1.2) should be accepted as natural ones.

Theorem *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that*

$$F(a,b)F(c,d) = F(ac,bd)$$

for all

$$a,b,c,d \in \mathbf{C}$$

and

$$F(a,b) = \frac{a}{b}, \quad a,b \in \mathbf{C}, b \neq 0.$$

Then, we obtain, for any $a \in \mathbf{C}$

$$F(a,0) = 0.$$

Proof. We have $F(a,0) = F(a,0)1 = F(a,0)\frac{2}{2} = F(a,0)F(2,2) = F(a \cdot 2, 0 \cdot 2) = F(2a,0) = F(2,1)F(a,0) = 2F(a,0)$.

Thus $F(a,0) = 2F(a,0)$ which implies the desired result $F(a,0) = 0$ for all $a \in \mathbf{C}$.

Several mathematicians pointed out to the author for the publication of the paper ([22]) that the notations of 100/0 and 0/0 are not good for the sake of the generalized senses, however, there does not exist other natural and good meaning for them. Why should we need and use any new notations for involving the notations? We should use the notation, we think so. Indeed, we will see in this book that many and many fractions in our formulas will have this meaning with the concept of the division by zero calculus for the case of functions.

2.3 By the Yamada field containing the division by zero

As an algebraic structure, we will give the simple filed structure containing the division by zero.

We consider

$$\mathbf{C}^2 = \mathbf{C} \times \mathbf{C}$$

and the direct decomposition

$$\mathbf{C}^2 = (\mathbf{C} \setminus \{0\})^2 + (\{0\} \times (\mathbf{C} \setminus \{0\})) + ((\mathbf{C} \setminus \{0\}) \times \{0\}) + \{0\}^2 .$$

Then, we note that

Theorem 1. *For the set* $\mathbf{C}^2$*, we introduce the relation* $\sim$*: for any* $(a, b), (c, d) \in (\mathbf{C} \setminus \{0\})^2$*,*

$$(a, b) \sim (c, d) \Longleftrightarrow ad = bc$$

and, for any $(a, b), (c, d) \notin (\mathbf{C} \setminus \{0\})^2$*, in the above direct decomposition*

$$(a, b) \sim (c, d).$$

Then, the relation $\sim$ *satisfies the equivalent relation.*

Definition 1. *For the quotiant set by the relation* $\sim$ *of the set* $\mathbf{C}^2$*, we write it by* A *and for the class containing* (a, b)*, we shall write it by* $\frac{a}{b}$*.*

Note that

Lemma 1. *In* $\mathbf{C}^2$*, if* $(a, b) \sim (m, n)$ *and* $(c, d) \sim (p, q)$*, then* $(ac, bd) \sim (mp, nq)$*.*

Then, we obtain the main result, as we can check easily:

Theorem 2. *For any members* $\frac{a}{b}, \frac{c}{d} \in A$*, we introduce the product* $\cdot$ *as follows:*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

and the sum $+$*:*

$$\frac{a}{b} + \frac{c}{d} = \begin{cases} \frac{c}{d}, & \text{if} \quad \frac{a}{b} = \frac{0}{1} \\ \frac{a}{b}, & \text{if} \quad \frac{c}{d} = \frac{0}{1}, \\ \frac{ad+bc}{bd}, & \text{if} \quad \frac{a}{b}, \frac{c}{d} \neq \frac{0}{1}, \end{cases}$$

then, the product and the sum are well-defined and A *becomes a field* $\mathbf{Y}$*.*

Indeed, we can see easily the followings: 1) Under the operation $+$, $\mathbf{Y}$ becomes an abelian group and $\frac{0}{1} = 0_Y$ is the unit element.

2) Under the operation $\cdot$, $\mathbf{Y} \setminus \{0_Y\}$ becomes an abelian group and $\frac{1}{1}$ is the unit element.

3) In $\mathbf{Y}$, operations $+$ and $\cdot$ satisfies distributive law.

Remark. *In* $\mathbf{C}^2$*, when* $(a,b) \sim (m,n)$ *and* $(c,d) \sim (p,q)$*, the relation* $(ad+bc, bd) \sim (mq+np, nq)$ *is, in general, not valid. In general,*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

is not well-defined and is not valid.

Indeed, $(1,2) \sim (1,2)$ and $(3,0) \sim (0,3)$, but

$$(1 \cdot 0 + 2 \cdot 3, 2 \cdot 0) = (6,0) \not\sim (3,6) = (1 \cdot 3 + 2 \cdot 0, 2 \cdot 3).$$

Theorem 3. *The two fields* $\mathbf{Y}$ *and* $\mathbf{C}$ *are homomorphic.*

Indeed, consider the mapping f from $\mathbf{Y}$ to $\mathbf{C}$:

$$f : x = \frac{a}{b} \mapsto \begin{cases} ab^{-1} & \left(\frac{a}{b} \neq 0_Y\right) \\ 0 & \left(\frac{a}{b} = 0_Y\right). \end{cases}$$

Then, we can see easily the followings: 1)$f(x+y) = f(x)+f(y)$, 2)$f(x \cdot y) = f(x)f(y)$, 3)$f(1_Y) = 1$, and 4)f is a one to one and onto mapping from $\mathbf{Y}$ to $\mathbf{C}$.

We define a unary operation φ_Y on $\mathbf{Y}$ as

$$\varphi_Y\left(\frac{a}{b}\right) = \frac{b}{a}.$$

For the inverse element of $x = \frac{a}{b} \neq 0_Y$, we shall denote by x^{-1}.

Definition 2. *We define a binary operation* $/$ *on* $\mathbf{Y}$ *as follows: For any* x, $y \in \mathbf{Y}$

$$x/y = x \cdot \varphi_Y(y) = \begin{cases} xy^{-1} & (y \neq 0_Y), \\ 0 & (y = 0_Y). \end{cases}$$

We will call the field $\mathbf{Y}$ with the operation φ_Y *0-divisible field*.

Theorem 3. **C** *becomes a 0-divisible field.*

Indeed, in **C**, a unary operation $\varphi = f \circ \varphi_Y \circ f^{-1}$ is induced by the homomorphic f from the 0-divisible field **C**. Then, for any $z \in \mathbf{C}$,

$$\varphi(z) = \begin{cases} z^{-1} & (z \neq 0)\,, \\ 0 & (z = 0)\,. \end{cases}$$

We, however, would like to state that *the division by zero* $z/0 = 0$ *is essentially, just the definition, and we can derive all the properties of the division by zero, essentially, from the definition.* Furthermore, by the idea of this session, we can introduce the fundamental concept of the divisions (fractions) in any field.

We should use the 0-divisible field **Y** for the complex numbers field **C** as complex numbers, by this simple modification.

Consider the lines $y = ax$ with gradients a through the origin $(0, 0)$ in the (x, y) plane. Consider the two limits that $a \quad (> 0)$ tends to $+\infty$ and $a \quad (< 0)$ tends to $-\infty$, respectively. As their limits, we see that the limiting lines are y axis, in a sense. Note that the gradient of the y axis is zero, and is not infinity. This example shows as in the graph of the function $y = f(x) = 1/x$ at $x = 0$ as $f(0) = 0$, that was introduced by the division by zero $1/0 = 0$ mathematically.

2.4 By the intuitive meaning of the fractions (division) by H. Michiwaki

We will introduce an another approach. The division b/a may be defined **independently of the product**. Indeed, in Japan, the division b/a ; b **raru** a (**jozan**) is defined as how many a exists in b, this idea comes from subtraction a repeatedly. (Meanwhile, product comes from addition). In Japanese language for "division", there exists such a concept independently of product. H. Michiwaki and his 6 years old daughter said for the result $100/0 = 0$ that the result is clear, from the meaning of the fractions independently of the concept of product and they said: $100/0 = 0$ does not mean that $100 = 0 \times 0$. Meanwhile, many mathematicians had a confusion for the result. Her understanding is reasonable and may be acceptable: $100/2 = 50$ will mean that we divide 100 by 2, then each will have 50. $100/10 = 10$ will mean that we divide 100 by 10, then each will have 10. $100/0 = 0$ will mean that we do not divide 100, and then nobody will have at all and so 0. Furthermore, she said then the rest is 100; that is, mathematically;

$$100 = 0 \cdot 0 + 100.$$

Now, all the mathematicians may accept the division by zero $100/0 = 0$ with natural feelings as a trivial one?

For simplicity, we shall consider the numbers on non-negative real numbers. We wish to define the division (or fraction) b/a following the usual procedure for its calculation, however, we have to take care for the division by zero: The first principle, for example, for $100/2$ we shall consider it as follows:

$$100 - 2 - 2 - 2-, ..., -2.$$

How may times can we subtract 2? At this case, it is 50 times and so, the fraction is 50. The second case, for example, for $3/2$ we shall consider it as follows:

$$3 - 2 = 1$$

and the rest (remainder) is 1, and for the rest 1, we multiple 10, then we consider similarly as follows:

$$10 - 2 - 2 - 2 - 2 - 2 = 0.$$

Therefore $10/2 = 5$ and so we define as follows:

$$\frac{3}{2} = 1 + 0.5 = 1.5.$$

By these procedures, for $a \neq 0$ we can define the fraction b/a, usually. Here we do not need the concept of product. Except the zero division, all the results for fractions are valid and accepted. Now, we shall consider the zero division, for example, $100/0$. Since

$$100 - 0 = 100,$$

that is, by the subtraction $100 - 0$, 100 does not decrease, so we can not say we subtract any from 100. Therefore, the subtract number should be understood as zero; that is,

$$\frac{100}{0} = 0.$$

We can understand this: the division by 0 means that it does not divide 100 and so, the result is 0. Similarly, we can see that

$$\frac{0}{0} = 0.$$

As a conclusion, we should define the zero division as, for any b

$$\frac{b}{0} = 0.$$

For complex numbers, we can consider the division $\frac{z_l}{z_2}$, similarly, by using the Euler formula

$$\frac{z_l}{z_2} = \frac{r_l}{r_2}\{\cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2)\} \tag{2.5}$$

for $|z_j| = r_j$ and $\arg z_j = \theta_j$. The problem may be reduced to one of the division $\frac{r_l}{r_2}$.

See [22] for the details.

2.5 Other introductions of general fractions

By the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$.

By considering the values of functions with the mean values of functions, we can introduce the general fractions. Note here that the Cauchy integral formula may be considered as a mean value theorem. The meanvalues will be considered as a fundamental concept in analysis.

On the division by zero in our theory, we will need only one new assumption in our mathematics that for the elementary function $W = 1/z$, $W(0) = 0$. However, for algebraic calculation of the division by zero, we have to follow the law of the Yamada field. For functions, however, we have to consider the concept of **the division by zero calculus**, as we will develop the details later with many applications.

3 Stereographic projection

For a great meaning and importance, we will see that the point at infinity is represented by zero.

3.1 The point at infinity is represented by zero

By considering the stereographic projection, we will be able to see that the point at infinity is represented by zero.

Consider the sphere (ξ, η, ζ) with radius $1/2$ put on the complex $z = x+iy$ plane with center $(0, 0, 1/2)$. From the north pole $N(0, 0, 1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the sphere onto the complex $z(= x + iy)$ plane; that is,

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}. \tag{3.1}$$

If $\zeta = 1$, then, by the division by zero, the north pole corresponds to the origin $(0, 0) = 0$.

Here, note that

$$x^2 + y^2 = \frac{\zeta}{1-\zeta}.$$

For $\zeta = 1$, we should consider as $1/0 = 0$, not by the division by zero calculus,

$$\frac{\zeta}{1-\zeta} = -1 - \frac{1}{\zeta - 1}.$$

We will consider the unit sphere $\{(x_1, x_2, x_3); x_1^2 + x_2^2 + x_3^2 = 1\}$. From the north pole $N(0, 0, 1)$, we consider the stereographic projection of the point $P(x_1, x_2, x_3)$ on the sphere onto the (x, y) plane; that is,

$$(x_1, x_2, x_3) = \tag{3.2}$$

$$\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1 - 1/(x^2+y^2)}{1 + 1/(x^2+y^2)}\right).$$

Then, we see that the north pole corresponds to the origin.

Next, we will consider the semi-sphere (ξ, η, ζ) with center $C(0, 0, 1)$ on the origin on the (x, y) plane. From the center $C(0, 0, 1)$, we consider the stereographic projection of the point $P(\xi, \eta, \zeta)$ on the semi- sphere onto the complex (x, y) plane; that is,

$$x = \frac{\xi}{1-\zeta}, y = \frac{\eta}{1-\zeta}. \tag{3.3}$$

If $\zeta = 1$, then, by the division by zero, the center C corresponds to the origin $(0, 0)$.

Meanwhile, we will consider the mapping from the open unit disc onto $\mathbf{R}^2$ in one to one and onto

$$\xi = \frac{x\sqrt{x^2+y^2}}{1+x^2+y^2}, \quad \eta = \frac{y\sqrt{x^2+y^2}}{1+x^2+y^2}$$

or

$$x = \frac{\xi}{\sqrt{\rho(1-\rho)}}, y = \frac{\eta}{\sqrt{\rho(1-\rho)}}; \quad \rho^2 = \xi^2 + \eta^2.$$

Note that the point $(x, y) = (0, 0)$ corresponds to $\rho = 0; (\xi, \eta) = (0, 0)$ and $\rho = 1$.

3.2 A contradiction of classical idea for $1/0 = \infty$

The infinity ∞ may be considered by the idea of the limiting, however, we had considered it as a number, for sometimes, typically, the point at infinity was represented by ∞ for some long years. For this fact, we will show a formal contradiction.

We will consider the stereographic projection by means of the unit sphere

$$\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = 1$$

from the complex $z = x + iy$ plane onto the sphere. Then, we obtain the correspondences

$$x = \frac{\xi}{1-\zeta}, \quad y = \frac{\eta}{1-\zeta}$$

and

$$\xi = \frac{1}{2}\frac{z+\overline{z}}{z\overline{z}+1}, \eta = \frac{1}{2i}\frac{z-\overline{z}}{z\overline{z}+1}, \zeta = \frac{z\overline{z}}{z\overline{z}+1}.$$

In general, two points P and Q_1 on the diameter of the unit sphere correspond to z and z_1, respectively if and only if

$$z\overline{z_1} + 1 = 0. \tag{3.4}$$

Meanwhile, two points P and Q_2 on the symmetric points on the unit sphere with respect to the plane $\zeta = \frac{1}{2}$ correspond to z and z_2, respectively if and only if

$$z\overline{z_2} - 1 = 0. \tag{3.5}$$

If the point P is the origin or the north pole, then the points Q_1 and Q_2 are the same point. Then, the identities (3.4) and (3.5) are not valid that show a contradiction.

Meanwhile, if we write (3.4) and (3.5)

$$z = -\frac{1}{\overline{z_1}} \tag{3.6}$$

and

$$z = \frac{1}{\overline{z_2}}, \tag{3.7}$$

respectively, we see that the division by zero (1.2) is valid.

3.3 Natural meanings of $1/0 = 0$

For constants a and b satisfying

$$\frac{1}{a} + \frac{1}{b} = k, \quad (\neq 0, const.)$$

the function

$$\frac{x}{a} + \frac{y}{b} = 1$$

passes the point $(1/k, 1/k)$. If $a = 0$, then, by the division by zero, $b = 1/k$ and $y = 1/k$; this result is natural.

We will consider the line $y = m(x-a)+b$ through a fixed point (a, b); $a, b > 0$ with gradient m. We set $A(0, -am+b)$ and $B(a-(b/m), 0)$ that are common points with the line and both lines $x = 0$ and $y = 0$, respectively. Then,

$$\overline{AB}^2 = (-am + b)^2 + \left(a - \frac{b}{m}\right)^2.$$

If $m = 0$, then $A(0, b)$ and $B(a, 0)$, by the division by zero, and furthermore

$$\overline{AB}^2 = a^2 + b^2.$$

Then, the line AB is a corresponding to the line between the origin and the point (a, b). Note that this line has only one common point with the both lines $x = 0$ and $y = 0$. Therefore, this result will be very natural in a sense. – Indeed, we can understand that the line $\overline{AB}$ is broken as the two lines $(0, b) - (a, b)$ and $(a, b) - (a, 0)$, suddenly.

The general line equation with gradient m is given by, with a constant b

$$y = m(x - a) + b \tag{3.8}$$

or

$$\frac{y}{m} = x - a + \frac{b}{m}. \tag{3.9}$$

By $m = 0$, we obtain the equation $x = a$, by the division by zero. This equation may be considered the cases $m = \infty$ and $m = -\infty$, and these cases may be considered by the strictly right logic with the division by zero.

By the division by zero, we can consider the equation (3.8) as a general line equation.

In the Lami's formula for three vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ satisfying

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}, \tag{3.10}$$

$$\frac{\|\mathbf{A}\|}{\sin\alpha} = \frac{\|\mathbf{B}\|}{\sin\beta} = \frac{\|\mathbf{C}\|}{\sin\gamma}, \tag{3.11}$$

if $\alpha = 0$, then we obtain:

$$\frac{\|\mathbf{A}\|}{0} = \frac{\|\mathbf{B}\|}{0} = \frac{\|\mathbf{C}\|}{0} = 0, \tag{3.12}$$

Here, of course, α is the angle of $\mathbf{B}$ and $\mathbf{C}$, β is the angle of $\mathbf{C}$ and $\mathbf{A}$, and γ is the angle of $\mathbf{A}$ and $\mathbf{B}$,

For the Newton's formula; that is, for a C^2 class function $y = f(x)$, the curvature K at the origin is given by

$$K = \lim_{x \to 0} \left| \frac{x^2}{2y} \right| = \left| \frac{1}{f''(0)} \right|, \tag{3.13}$$

we have: for $f''(0) = 0$,

$$K = \frac{1}{0} = 0. \tag{3.14}$$

Recall the formula

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin nx dx = -\frac{2}{n},$$

for

$$n = \pm 1, \pm 2, ..., ...$$

Then, for $n = 0$, we have

$$b_0 = -\frac{2}{0} = 0.$$

3.4 Double natures of the zero point $z = 0$

Any line on the complex plane arrives at the point at infinity and the point at infinity is represented by zero. That is, a line is, indeed, contains the origin; the true line should be considered as the sum of a usual line and the origin. We can say that it is a compactification of the line and the compacted point is the point at infinity, however, it it is represented by $z = 0$. Later, we will see this property by analytic geometry and the division by zero calculus in many situations.

However, for the general line equation

$$ax + by + c = 0, \tag{3.15}$$

by using the polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have

$$r = \frac{-c}{a\cos\theta + b\sin\theta}. \tag{3.16}$$

When $a\cos\theta + b\sin\theta = 0$, by the division by zero, we have $r = 0$; that is, we can consider that the line contains the origin.

The envelop of the linear lines represented by, for constants m and a fixed constant $p > 0$,

$$y = mx + \frac{p}{m}, \tag{3.17}$$

we have the function, by using an elementary ordinary differential equation,

$$y^2 = 4px. \tag{3.18}$$

The origin of this parabolic function is missing from the envelop of the linear functions, because the linear equations do not contain the y axis as the tangential line of the parabolic function. Now recall that, by the division by zero, as the linear equation for $m = 0$, we have the function $y = 0$, the x axis. Note that both the x axis $y = 0$ and the parabolic function have the zero gradient at the origin; that will mean that in the reasonable sense the x axis is a tangential line of the parabolic function. Anyhow, by the division by zero, the envelop of the linear functions may be considered as the whole parabolic function containing the origin.

When we consider the limiting of the linear equations as $m \to 0$, we will think that the limit function is a parallel line to the x axis through the point at infinity. Since the point at infinity is represented by zero, it will become the x axis.

Meanwhile, when we consider the limiting function as $m \to \infty$, we have the y axis $x = 0$ and this function is an ordinally tangential line of the parabolic function. From these two tangential lines, we see that the origin has **double natures**; one is the continuous tangential line $x = 0$ and the second is the discontinuous tangential line $y = 0$.

In addition, note that the tangential point of (3.18) for the line (3.17) is given by

$$\left(\frac{p}{m}, \frac{2p}{m} \right) \tag{3.19}$$

and it is $(0, 0)$ for $m = 0$.

We can see the point at infinity is reflected to the origin; and so, the origin has the double natures; one is the native origin and another is reflected to the origin of the point at infinity.

4 Mirror image with respect to a circle

For simplicity, we will consider the unit circle $|z| = 1$ on the complex $z = x + iy$ plane. Then, we have the reflection formula

$$z^* = \frac{1}{\overline{z}} \tag{4.1}$$

for any point z, as well-known ([2]). For the reflection point z^*, there is no problem for the points $z \neq 0, \infty$. As the classical result, the reflection of zero is the point at infinity and conversely, for the point at infinity we have the zero point. The reflection is a one to one and onto mapping between the inside and the outside of the unit circle, by considering the point at infinity.

Are these correspondences, however, suitable? Does there exist the point at ∞, really? Is the point at infinity corresponding to the zero point, by the reflection? Is the point at ∞ reasonable from the practical point of view? Indeed, where can we find the point at infinity? Of course, we know plesantly the point at infinity on the Riemann sphere, however, on the complex z-plane it seems that we can not find the corresponding point. When we approach to the origin on a radial line, it seems that the correspondence reflection points approach to *the point at infinity* with the direction (on the radial line).

On the concept of the division by zero, there is no the point at infinity ∞ as the numbers. For any point z such that $|z| > 1$, there exists the unique point z^* by (4.1). Meanwhile, for any point z such that $|z| < 1$ except $z = 0$, there exits the unique point z^* by (4.1). Here, note that for $z = 0$, by the division by zero, $z^* = 0$. Furthermore, we can see that

$$\lim_{z \to 0} z^* = \infty, \tag{4.2}$$

however, for $z = 0$ itself, by the division by zero, we have $z^* = 0$. This will mean a strong discontinuity of the functions $W = \frac{1}{z}$ and (4.1) at the origin $z = 0$; that is a typical property of the division by zero. This strong discontinuity may be looked in the above reflection property, physically.

The result is a surprising one in a sense; indeed, by considering the geometrical corresponding of the mirror image, we will consider the center corresponds to the point at infinity that is represented by the origin $z = 0$. This will show that the mirror image is not followed by this concept; the corresponding seems to come from the concept of one-to-one and onto mapping.

Should we exclude the point at infinity, from the numbers? We were able to look the strong discontinuity of the division by zero in the reflection with respect to circles, physically (geometrical optics). The division by zero gives a one to one and onto mapping of the reflection (4.1) from the whole complex plane onto the whole complex plane.

The infinity ∞ may be considered as in (4.2) as the usual sense of limits, however, the infinity ∞ is not a definite number.

We consider a circle on the complex z plane with center z_0 and radius r. Then, the mirror image relation p and q with respect to the circle is given by

$$p = z_0 + \frac{r^2}{\overline{q - z_0}}. \tag{4.3}$$

For $q = z_0$, we have, by the division by zero,

$$p = z_0, \tag{4.4}$$

For a circle

$$Az\overline{z} + \overline{\beta}z + \beta\overline{z} + D = 0; \quad A > 0, D : real \quad number, \tag{4.5}$$

or

$$\left(z + \frac{\beta}{A}\right)\overline{\left(z + \frac{\beta}{A}\right)} = \frac{|\beta|^2 - AD}{A^2}, \tag{4.6}$$

the points z and z_1 are in the relation of the mirror image with respect to the circle if and only if

$$Az_1\overline{z} + \overline{\beta}z_1 + \beta\overline{z} + D = 0, \tag{4.7}$$

or

$$\overline{z_1} = -\frac{\overline{\beta}z + D}{Az + \beta} \tag{4.8}$$

$$= -\frac{\overline{\beta}}{A} - \frac{1}{A}\left(D - \frac{|\beta|^2}{A}\right)\frac{1}{z - \left(-\frac{\beta}{A}\right)}.$$

The center $-\beta/A$ corresponds to the center itself, as we see from the division by zero for (4.6).

On the x, y plane, we shall consider the inversion relation with respect to the circle with radius R and with center at the origin:

$$x' = \frac{xR^2}{x^2 + y^2}, \quad y' = \frac{yR^2}{x^2 + y^2}. \tag{4.9}$$

Then, the line

$$ax + by + c = 0 \tag{4.10}$$

is transformed to the line

$$R^2(ax' + by') + c((x')^2 + (y')^2) = 0. \tag{4.11}$$

In particular, for $c = 0$, the line $ax + by = 0$ is transformed to the line $ax' + by' = 0$. This corresponding is one-to-one and onto, and so the origin $(0, 0)$ have to correspond to the origin $(0, 0)$.

For the elliptic curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0 \tag{4.12}$$

and for the similar correspondences

$$x' = \frac{a^2 b^2 x}{b^2 x^2 + a^2 y^2}, \quad y' = \frac{a^2 b^2 y^2}{b^2 x^2 + a^2 y^2}. \tag{4.13}$$

the origin corresponds to itself.

The pole (x_1, y_1) of the line

$$ax + by + c = 0 \tag{4.14}$$

with respect to a circle with radius R with center (x_0, y_0) is given by

$$x_1 = x_0 - \frac{aR^2}{ax_0 + by_0 + c} \tag{4.15}$$

and

$$y_1 = y_0 - \frac{bR^2}{ax_0 + by_0 + c}. \tag{4.16}$$

If $ax_0 + by_0 + c = 0$, then we have $(x_1, y_1) = (x_0, y_0)$.

Furthermore, for various higher dimensional cases the results are similar.

5 Division by zero calculus

As the number system containing the division by zero, the Yamada field structure is complete.

However for applications of the division by zero to **functions**, we will need the concept of division by zero calculus for the sake of uniquely determinations of the results and for other reasons. See [27].

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i}, \tag{5.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)}, \tag{5.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i}, \tag{5.3}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=i} \cdot \left[\frac{1}{z+i} \right|_{z=i} = 0 \cdot (-2i) = 0. \tag{5.4}$$

The short version of this section was given by [42] in the Proceedings of the Internationa Conference: https://sites.google.com/site/sandrapinelas/icddea-2017

5.1 Introduction of the division by zero calculus

Therefore, we will introduce the division by zero calculus: For any formal Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z-a)^n \tag{5.5}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{5.6}$$

Note that here, there is no problem on any convergence of the expansion (5.5) at the point $z = a$.

For the correspondence (5.6) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (5.5), we can define any order derivatives of the function f at the singular point a.

In order to avoid any logical confusion in the division by zero, we would like to refer to the logical essence:

For the elementary function $W = f(z) = 1/z$, we define $f(0) = 0$ and we will write it by $1/0$ following the form, apart from the sense of the intuitive sense of fraction. With only this new definition, we can develop our mathematics, through the division by zero calculus.

As a logical line for the division by zero, we can consider as follows:

We define $1/0 = 0$ for the form; this precise meaning is that for the function $W = f(z) = 1/z$, we have $f(0) = 0$ following the form. Then, we can define the division by zero calculus (5.6) for (5.5). In particular, from the function $f(x) \equiv 0$ we have $0/0 = 0$. In this sense, $1/0 = 0$ is more fundamental than $0/0 = 0$; that is, from $1/0 = 0$, $0/0 = 0$ is derived.

We will give typical and various examples.

For the typical function $\sin x/x$, we have

$$\frac{\sin x}{x}(0) = \frac{\sin 0}{0} = \frac{0}{0} = 0,$$

however, by the division by zero calculus, we have, for the function $(\sin x)/x$

$$\frac{\sin x}{x}(0) = 1,$$

that is more reasonable in analysis.

However, for functions we see that the results by the division by zero calculus have not always practical senses and so, **for the results by division by zero we should check the results, case by case.**

For example, for the simple example for the line equation on the x, y plane

$$ax + by + c = 0$$

we have, formally

$$x + \frac{by + c}{a} = 0,$$

and so, by the division by zero, we have, for $a = 0$, the reasonable result

$$x = 0.$$

Indeed, for the equation $y = mx$, from

$$\frac{y}{m} = x,$$

we have, by the division by zero, $x = 0$ for $m = 0$. This gives the case $m = \pm\infty$ of the gradient of the line. – This will mean that the equation $y = mx$ represents the general line through the origin in this sense. – This method was applied in many cases, for example see [36, 37].

However, from

$$\frac{ax + by}{c} + 1 = 0,$$

for $c = 0$, we have the contradiction, by the division by zero

$$1 = 0.$$

Meanwhile, note that for the function $f(z) = z + \frac{1}{z}$, $f(0) = 0$, however, for the function

$$f(z)^2 = z^2 + 2 + \frac{1}{z^2},$$

we have $f^2(0) = 2$. Of course,

$$f(0) \cdot f(0) = \{f(0)\}^2 = 0.$$

In the formula

$$\frac{x^{a+1}}{a+1}\left(\log x - \frac{1}{x+1}\right),$$

for $a = -1$, we have, by the division by zero calculus,

$$\frac{1}{2}\left(\log x\right).$$

Furthermore, see many examples, [27].

For a smooth function $f(x)$ of class C^n for $n \geq 1$, from the Taylor expansion around $x = a$, we have:

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(c)}{n!}(x-a)^n. \tag{5.7}$$

$$a < c < x \quad or \quad a > c > x.$$

Then, we obtain, by the division by zero

$$\left[\frac{f(x)}{(x-a)^m}\right]_{x=a} = \begin{cases} 0 \quad (m > n) \\ \frac{f^{(m)}(a)}{m!} \quad (m \leq n). \end{cases} \tag{5.8}$$

Note that the division by zero calculus was defined by the value of the function at the point a, not by the limiting $x \to a$. Therefore, the function $\frac{f^{(n)}(c)}{n!}$ in (5.7) is $\frac{f^{(n)}(a)}{n!}$ at the point a. The division by zero calculus is defined for analytic functions at isolated singular points by using the Laurent expansion, but for smooth functions that are not analytic, we will be able to consider the division by zero calculus by this sense, by using the Taylor expansion.

Note, in particular, that for a function $f(x)$ of class C^2 around $x = a$, by the division by zero,

$$\left[\frac{f(a+h) + f(a-h) - 2f(a)}{h^2}\right]_{h=0} = f''(a). \tag{5.9}$$

For the function

$$f(x) = x \sin\frac{1}{x},$$

if $f(0) = 0$, then the function is continuous at $x = 0$, however, it is not differentiable at the origin. By the division by zero calculus, we have, automatically

$$f(0) = 1.$$

Meanwhile, we have an interesting formula whose proof is simple:

Theorem 1. *Consider a family of absolutely continuous functions* $f_a(x)$ *that is analytic in* $a \in \mathbb{R} \setminus \{a_0\}$. *Let* $g_a(x) = f'_a(x)$, *and we assume that it is extensible to the point on* $a = a_0$ *as an absolutely continuous function, then*

$$f_{a_0}(x) = \int g_{a_0}(x)\,dx.$$

We will show examples:

1. Let $f_n(x) = \frac{(ax+b)^{n+1}}{a(n+1)}$ where $a \in \mathbb{R} \setminus \{0\}$ and $n+1$: poisitive integers. Then $g_n(x) = (ax+b)^n$ and

$$\left[\frac{(ax+b)^{n+1}}{a\,(n+1)}\right]_{n=-1} = \int (ax+b)^{-1}\, dx = \frac{\ln|ax+b|}{a}, \ a \neq 0;$$

by the same way we have

$$\left[\frac{(ax+b)^{n+1}}{a\,(n+1)}\right]_{a=0} = \int b^n dx = b^n x.$$

2. Let $f(x) = \frac{\arctan(x/a)}{a}$ where $a \in \mathbb{R} \setminus \{0\}$. On this case we get $g_n(x) = \frac{1}{x^2+a^2}$ and consequently

$$\left[\frac{\arctan(x/a)}{a}\right]_{a=0} = \int \frac{1}{x^2} dx = -\frac{1}{x}.$$

3. Let $f(x) = \frac{a^x}{\log a}, a > 0$. Then, we obtain

$$\left[\frac{a^x}{\log a}\right]_{a=1} = \int dx = x.$$

In this example, note that the function $f(x)$ may not be considered in the sense of the Laurent expansion in a. However, by setting $\log a = A$, we can obtain that:

$$\frac{e^{Ax}}{A}|_{A=0} = x,$$

by the division by zero calculus. In the formula

$$\int a^x dx = \frac{a^x}{\log a} + C,$$

for $a = 1$, the formula

$$\int 1^x dx = \frac{1^x}{\log 1} + C$$

is not valid.

We, meanwhile, obtain that

$$\left(\frac{1}{\log x}\right)_{x=1} = 0. \tag{5.10}$$

Indeed, we consider the function $y = \exp(1/x), x \in \mathbf{R}$ and its inverse function $y = \frac{1}{\log x}$. By the symmetric property of the two functions with respect to the function $y = x$, we have the desired result.

Here, note that for the function $\frac{1}{\log x}$, we cannot use the Laurent expansion around $x = 1$, and therefore, the result is not trivial.

Theorem 2. *Consider a family of absolutely continuous functions $F_a(x)$ where $a \in I \subset R$, I is an open interval, and*

$$f_a(x) = \int F_a(x) dx.$$

If a point a_0 is a pole of order n of the analytic functions $f_a(x)$ as functions in a and there exists an analytic function $g : I \to \mathbb{R}$ for any fixed x such that $g(a,x) = (a - a_0)^n f_a(x)$ then

$$\int F_{a_0}(x) dx = \frac{g^{(n)}(a_0, x)}{n!}.$$

Proof: Using the Taylor theorem, we have, for any fixed x

$$g(a,x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(a_0, x)}{k!} (a - a_0)^k,$$

and by the division by zero calculus, we have

$$\int F_{a_0}(x) dx = f_{a_0}(x) = \left[\frac{1}{(a - a_0)^n} g(a,x)\right]_{a=a_0} = \frac{g^{(n)}(a_0, x)}{n!}.$$

Theorems 1 and 2 were discovered by S. Pinelas (see [42]).
We shall give examples.

1. For the integral

$$\int x(x^2+1)^a dx = \frac{(x^2+1)^{a+1}}{2(a+1)} \quad (a \neq -1), \tag{5.11}$$

   we obtain, by the division by zero,

$$\int x(x^2+1)^{-1} dx = \frac{\log(x^2+1)}{2}. \tag{5.12}$$

2. For the integral

$$\int \sin ax \cos x dx = \frac{\sin ax \sin x + a \cos ax \cos x}{1-a^2} \quad (a^2 \neq 1), \tag{5.13}$$

   we obtain, by the division by zero, for the case $a = 1$

$$\int \sin x \cos x dx = \frac{\sin^2 x}{2} - \frac{1}{4}. \tag{5.14}$$

3. For the integral

$$\int \sin^{\alpha-1} x \cos(\alpha+1)x dx = \frac{1}{\alpha} \sin^{\alpha} x \cos \alpha x, \tag{5.15}$$

   we obtain, by the division by zero, for the case $\alpha = 0$

$$\int \sin^{-1} x \cos x dx = \log \sin x. \tag{5.16}$$

4. For the integral

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2} \left(a \sin bx + b \cos bx \right), \tag{5.17}$$

   for example, we can consider the case $a = bi$, by the division by zero calculus, and we can obtain the expected good result.

We can obtain many and many such identities.

We will state the formal theorem whoes proof is trivial:

Theorem 3. *Consider an operator L that transforms functions $f_z(t)$ on a set T with analytic parameter z of an isolated singular point a into functions $L[f_z(t)] = F_z(s)$ on a set S. Assume that for the Laurent expansions around a point $a \in D$, a disc on the complex z plane with center a, for any fixed t*

$$f_z(t) = \sum_{n=-\infty}^{\infty} f_n(t)(z-a)^n,$$

$$L[f_z(t)] = \sum_{n=-\infty}^{\infty} L[f_n(t)](z-a)^n.$$

Then we have

$$F_a(s) = L[f_0(t)]. \tag{5.18}$$

We illustrate this result with examples:

1. Let $f_\lambda(t) = \frac{\sin(\lambda t)}{\lambda}$, where $\lambda \in \mathbb{R} \setminus \{0\}$. The Laplace transform of $f_\lambda(t)$ is

$$L\left[\frac{\sin(\lambda t)}{\lambda}\right] = \frac{1}{s^2+\lambda^2}$$

for $\lambda \neq 0$. Then we have:

$$L[t] = \frac{1}{s^2}.$$

2. Let $f_{\mu,\lambda}(t) = \frac{e^{\mu t}-e^{\lambda t}}{\mu-\lambda}$, where $\mu \neq \lambda$. The Laplace transform of $f_{\mu,\lambda}(t)$ is

$$L\left[\frac{e^{\mu t}-e^{\lambda t}}{\mu-\lambda}\right] = \frac{1}{(s-\mu)(s-\lambda)}$$

for $\mu \neq \lambda$. Then we have:

$$L\left[te^{\lambda t}\right] = \frac{1}{(s-\lambda)^2}. \tag{5.19}$$

3. We consider the function

$$f(t) = \begin{cases} 2t, & \text{if} \quad 0 \le t < 1; \\ 3 - t, & \text{if} \quad 1 \le t < 2; \\ 0, & \text{if} \quad t \ge 2, \end{cases}$$

whose Laplace transform is

$$F(s) = \frac{1 - 2e^{-s} + e^{-3s}}{s^2} \quad (s > 0).$$

([50]). Then, by l'Hopital law, we can not derive the value at $s = 0$ as $7/2$, which is derived by the division by zero calculus.

4. As a typical example in A. Kaneko ([21], page 11) in the theory of hyperfunction theory: for non-integers λ, we have

$$x_+^\lambda = \left[\frac{-(-z)^\lambda}{2i \sin \pi\lambda}\right] = \frac{1}{2i \sin \pi\lambda}\{(-x + i0)^\lambda - (-x - i0)^\lambda\} \tag{5.20}$$

where the left hand side is a Sato hyperfunction and the middle term is the representative analytic function whose meaning is given by the last term. For an integer n, Kaneko derived that

$$x_+^n = \left[-\frac{z^n}{2\pi i} \log(-z)\right], \tag{5.21}$$

where log is a principal value: $\{-\pi < \arg z < +\pi\}$. Kaneko stated there that by taking a finite part of the Laurent expansion, the formula is derived. Indeed, we have the expansion, for around n, integer

$$\frac{-(-z)^\lambda}{2i \sin \pi\lambda}$$

$$= \frac{-z^n}{2\pi i} \frac{1}{\lambda - n} - \frac{z^n}{2\pi i} \log(-z) - \left(\frac{\log^2(-z) z^n}{2\pi i \cdot 2!} + \frac{\pi z^n}{2i \cdot 3!}\right)(\lambda - n) + ... \tag{5.22}$$

([21], page 220).

By Theorem 3, however, we can derive this result (5.21) from the Laurent expansion (5.22), immediately.

Meanwhile, M. Morimoto derived this result by using the Gamma function with the elementary means in [29], pages 60-62. See also [17].

5. For many generating functions we can obtain some interesting identities. For example, we will consider the mapping

$$\zeta \in C \setminus \{0\} \to F(z,\zeta) := \exp \frac{z}{2}\left(\zeta - \frac{1}{\zeta}\right)$$

Then, from

$$F(z,\zeta) = \sum_{n=-\infty}^{+\infty} J_n(z)\zeta^n, \tag{5.23}$$

we obtain:

$$F(z,0) = J_0(z).$$

5.2 Difficulty in Maple for specialization problems

For the Fourier coefficients a_n

$$a_n = \int t \cos n\pi t dt = \frac{\cos n\pi t}{n^2\pi^2} + \frac{t}{n\pi}\cos n\pi t, \tag{5.24}$$

we obtain, by the division by zero calculus,

$$a_0 = \frac{t^2}{2}. \tag{5.25}$$

Similarly, for the Fourier coefficients a_n

$$a_n = \int t^2 \cos n\pi t dt = \frac{2t}{\pi^2 n^2}\cos n\pi t - \frac{2}{n^3\pi^3}\sin n\pi t + \frac{t^2}{n\pi}\sin n\pi t, \tag{5.26}$$

we obtain

$$a_0 = \frac{t^3}{3}. \tag{5.27}$$

For the Fourier coefficients a_k of a function :

$$\frac{a_k \pi k^3}{4}$$

$$= \sin(\pi k)\cos(\pi k) + 2k^2\pi^2 \sin(\pi k)\cos(\pi k) + 2\pi(\cos(\pi k))^2 - \pi k, \tag{5.28}$$

for $k = 0$, we obtain, by the division by zero calculation, immediately

$$a_0 = \frac{8}{3}\pi^2 \tag{5.29}$$

(see [59], (3.4)).

We have many such examples.

5.3 Reproducing kernels

The function

$$K_{a,b}(x,y) = \frac{1}{2ab} \exp\left(-\frac{b}{a}|x-y|\right)$$

is the reprodiucing kernel for the space $H_{K_{a,b}}$ equipped with the norm

$$\|f\|^2_{H_{K_{a,b}}} = \int (a^2 f'(x)^2 + b^2 f(x)^2) dx$$

([50], pages 15-16). If $b = 0$, then

$$K_{a,0}(x,y) = -\frac{1}{2a^2}|x-y|$$

is the reproducing kernel for the space $H_{K_{a,0}}$ equipped with the norm

$$\|f\|^2_{H_{K_{a,b}}} = a^2 \int (f'(x)^2 dx.$$

Meanwhile, if $a = 0, K_{0,b}(x,y) = 0$, then it is the trivial reproducing kernel for the zero function space.

We denote by $\mathcal{O}(\{0\})$ the set of all analytic functions defined on a neighborhood of the origin.

Then, we have:

Let $\{C_j\}_{j=0}^{\infty}$ be a positive sequence such that

$$\limsup_{j\to\infty} \sqrt[j]{C_j} < \infty. \tag{5.30}$$

Set

$$R \equiv \left(\limsup_{j\to\infty} \sqrt[j]{C_j}\right)^{-1} > 0 \tag{5.31}$$

and define a kernel K by

$$K(z,u) \equiv \sum_{j=0}^{\infty} C_j\, z^j\, \overline{u}^j \quad (|z|, |u| < \sqrt{R}). \tag{5.32}$$

Then we have

$$H_K(\Delta(\sqrt{R})) = \left\{ f \in \mathcal{O}(\Delta(\sqrt{R})) \ : \ \sqrt{\sum_{j=0}^{\infty} \frac{|f^{(j)}(0)|^2}{(j!)^2 C_j}} < \infty \right\} \tag{5.33}$$

and the norm is given by:

$$\|f\|_{H_K(\Delta(\sqrt{R}))} = \sqrt{\sum_{j=0}^{\infty} \frac{|f^{(j)}(0)|^2}{(j!)^2 C_j}}$$

that is the reproducing kernel Hilbert space admitting the kernel (5.32) ([50], page 35).

If some constant $C_{j_0} = 0$, then there is no problem, by interpretating that in the above statement

$$\frac{|f^{(j_0)}(0)|^2}{(j!)^2 C_{j_0}} = 0. \tag{5.34}$$

5.4 Ratio

On the real $x-$ line, we fix two different point $P_1(x_1)$ and $P_2(x_2)$ and we will consider the point, with a real number

$$P(x; r) = \frac{x_1 + r x_2}{1 + r}. \tag{5.35}$$

If $r = 1$, then the point $P(x; 1)$ is the mid point of the two points P_1 and P_2 and for $r > 0$, the point P is on the interval (x_1, x_2). Meanwhile, for $-1 < r < 0$, the point P is on $(-\infty, x_1)$ and for $r < -1$, the point P is on $(x_2, +\infty)$. Of course, for $r = 0$, $P = P_1$. We see that r tends to $+\infty$ and $-\infty$, P tends to the point P_2. We see the pleasant fact that by the division by zero calculus, $P(x, -1) = P_2$. For this fact we see that for all real numbers r correspond to all real line numbers.

In particular, we see that in many text books on the undergraduate course the formula (5.35) is stated as a parameter representation of the line through the two pints P_1 and P_2. However, if we do not consider the case $r = -1$ by the division by zero calculus, the classical statement is not right, because the point P_2 may not be considered.

On this setting, we will consider another representation

$$P(x; m, n) = \frac{mx_2 - nx_1}{m - n}$$

for the exterior division point $P(x; m, n)$ in $m : n$ for the point P_1 and P_2. For $m = n$. we obtain, by the division by zero calculus, $P(x; m, m) = x_2$. Imagine the result that the point $P(x; m, m) = P_2$ and the point P_2 seems to be the point P_1. Such a strong discontinuity happens for many cases. See [27, 35].

By the division by zero, we can introduce the ratio for any complex numbers a, b, c, d as

$$\frac{AC}{CB} = \frac{c - a}{b - c}. \tag{5.36}$$

We will consider the **Appollonius circle** determined by the equation

$$\frac{AP}{PB} = \frac{|z - a|}{|b - z|} = \frac{m}{n} \tag{5.37}$$

for fixed $m, n \geq 0$. Then, we obtain the equation for the cirlce

$$\left| z - \frac{-n^2 a + m^2 b}{m^2 - n^2} \right|^2 = \frac{m^2 n^2}{(m^2 - n^2)^2} \cdot |b - a|^2. \tag{5.38}$$

If $m = n \neq 0$, the circle is the line in (5.37). For $|m| + |n| \neq 0$, if $m = 0$, then $z = a$ and if $n = 0$, then $z = b$. If $m = n = 0$ then z is a or b.

The representation (5.37) is valid always, however, (5.38) is not reasonable for $m = n$. The property of the division by zero depends on the representations of formulas.

On the real line, the points $P(p), Q(1), R(r), S(-1)$ is a harmonic series if and only if

$$p = \frac{1}{r}.$$

If $r = 0$, then we have $p = 0$ that is now the representation of the point at infinity. (H. Okumura: 2017.12.27.)

5.5 Identities

For example, we have the identity

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{1}{(c-b)(a-c)(x-a)}$$

$$+\frac{1}{(b-c)(b-a)(x-b)} + \frac{1}{(c-a)(c-b)(x-c)}.$$

By the division by zero calculus, the first term in the right hand side is zero for $x = a$, and

$$\frac{1}{(b-c)(b-a)(a-b)} + \frac{1}{(c-a)(c-b)(a-c)}.$$

This result is the same as

$$\frac{1}{(x-a)(x-b)(x-c)}(a),$$

by the divison by zero calculus.

For the identity

$$\frac{1}{x(a+x)^2} = \frac{1}{a^2x} - \frac{1}{a(a+x)^2} - \frac{1}{a^2(a+x)}, \tag{5.39}$$

we have the identity for the both $\frac{1}{x^3}$.

For the identity

$$f(z) = \Pi_{j=1}^{n}(z - z_j), \tag{5.40}$$

we have the identity

$$\left[\frac{f'(z)}{f(z)}\right]_{z=z_1} = \frac{1}{z_1 - z_2} + ... + \frac{1}{z_1 - z_n}. \tag{5.41}$$

For the identity

$$\frac{mx+n}{ax^2+2bx+c} \tag{5.42}$$

$$= \frac{m}{2a}\frac{2ax+2b}{ax^2+2bx+c} + \frac{an-bm}{a} - \frac{1}{ax^2+2bx+c},$$

for $a = 0$, we have

$$\frac{mx+n}{2bx+c} = \frac{x(bx+c)}{(2bx+c)^2} + \frac{2bnx+nc+bmx^2}{(2bx+c)^2}. \tag{5.43}$$

For the identity

$$I_n = (-1)^n n! \frac{1}{(a^2+x^2)^{(n+1)/2}} \sin(n+1)\theta \tag{5.44}$$

$$= \frac{(-1)^n n!}{2i}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right], z = x+iy = e^{i\theta},$$

we have, for $x = ai$

$$[I_n]_{x=ai} = \frac{(-1)^n n!}{2^{n+2} i^n}. \tag{5.45}$$

In the identity, for $-\pi \leq x \leq \pi$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nx}{n^2 - a^2} = \frac{\pi \cos ax}{2a \sin a\pi} - \frac{1}{2a^2}, \tag{5.46}$$

for $a = 0$, we have

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nx}{n^2} = \frac{1}{12}(\pi^2 - 3x^2). \tag{5.47}$$

In the identity, for $0 < x < 2\pi, |a| \leq 1$

$$\sum_{n=1}^{\infty} a^{2n-1} \frac{\sin[(2n-1)x]}{2n-1} = \frac{1}{2}\tan^{-1}\frac{21 \sin x}{1-a^2}, \tag{5.48}$$

for $a = 1$, we have, for $0 < x < \pi$,

$$\sum_{n=1}^{\infty} \frac{\sin[(2n-1)x]}{2n-1} = \frac{\pi}{4}. \tag{5.49}$$

For the identities, for $0 \leq x \leq 2\pi$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}\cos(nx) = \frac{\pi}{2a\sinh(a\pi)}\cosh[a(\pi - x]] - \frac{1}{2a^2}, \tag{5.50}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2} \cos(nx) = \frac{\pi}{2a \sin(a\pi)} \cos[a(\pi - x]] + \frac{1}{2a^2}, \tag{5.51}$$

for $a = 0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) = \frac{1}{12}(3x^2 - 6\pi x + 2\pi^2). \tag{5.52}$$

We can derive many these type identities.

5.6 Remarks for the applications of the division by zero and the division by zero calculus

As the number system, we can calculus the arithmetic by the Yamada field structure. However, for functions, the problems are involved for their structures and we have also the delicate problems for the smoothness of functions. So, by applying the division by zero, we should consider and apply the division by zero and division by zero calculus in many ways and check the results. By considering many ways, we will be able to see many new aspects and results. By checking the results obtained, we will be able to find new prospects. With this idea, we can enjoy the division by zero calculus with free spirits without logical problems. – In this idea, we may ask what is mathematics?

6 Triangles and division by zero

In order to see how elementary of the division by zero, we will see the division by zero in triangles as the fundamental objects.

We will consider a triangle ABC with length a, b, c. Let θ be the angle of the side BC and the bisector line of A. Then, we have the identity

$$\tan\theta = \frac{c+b}{c-b}\tan\frac{A}{2}, \quad b < c.$$

For $c = b$, we have

$$\tan\theta = \frac{2b}{0}\frac{A}{2}.$$

Of course, $\theta = \pi/2$.

We have the formula

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\tan B}{\tan C}.$$

If $a^2+b^2-c^2 = 0$, then $C = \pi/2$. Then,

$$0 = \frac{\tan B}{\tan\frac{\pi}{2}} = \frac{\tan B}{0}.$$

Meanwhile, for the case $a^2-b^2+c^2 = 0$, then $B = \pi/2$, and we have

$$\frac{a^2+b^2-c^2}{0} = \frac{\tan\frac{\pi}{2}}{\tan C} = 0.$$

Meanwhile, the lengths f and f' of the bisector lines of A and in the out of the triangle ABC are given by

$$f = \frac{2bc\cos\frac{A}{2}}{b+c}$$

and

$$f' = \frac{2bc\sin\frac{A}{2}}{b-c},$$

respectively.

If $b = c$, then we have $f' = 0$, by the division by zero. However, note that, from

$$f' = 2\sin\frac{A}{2}\left(ac + \frac{c^2}{b-c}\right),$$

by the division by zero calculus, for $b = c$, we have

$$2b \sin \frac{A}{2} = a.$$

Let H be the perpendicular leg of A to the side BC and let E and M be the mid points of AH and BC, respectively. Let θ be the angle of EMB ($b > c$). Then, we have

$$\frac{1}{\tan \theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B = C$, then $\theta = \pi/2$ and $\tan(\pi/2) = 0$.

In the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

if b or c is zero, then, by the division by zero, we have $\cos A = 0$. Therefore, then we should understand as $A = \pi/2$.

Let r be the radius of the inscribed circle of the triangle ABC, and h_A, h_B, h_C be the distances from A,B,C to the lines BC, CA, AB, respectively. Then we have

$$\frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}.$$

When the point A is the point at infinity, then, $r_A = 0$ and $h_B = h_C = 2r$ and the identity still holds.

We have the identities

$$S = \frac{ah_A}{2} = \frac{1}{2} bc \sin A$$

$$= \frac{1}{2} a^2 \frac{\sin B \sin C}{\sin A}$$

$$= \frac{abc}{4R} = 2R^2 \sin A \sin B \sin C = rs, \quad s = \frac{1}{2}(a + b + c).$$

If A is the point at infinity, then, $S = s = h_A = b = c = 0$ and the above identities all valid.

For the identity

$$\tan \frac{A}{2} = \frac{r}{s - a},$$

if the point A is the point at infinity, $A = 0, s - a = 0$ and the identity holds as $0 = r/0$. Meanwhile, if $A = \pi$, then the identity holds as $\tan(\pi/2) = 0 = 0/s$.

If we write the triangle by vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$, the center vector $\mathbf{I}$ is given by

$$\mathbf{I} = \frac{1}{a+b+c}(a\mathbf{A} + b\mathbf{B} + c\mathbf{C}).$$

If the point A is the point at infinity, then $b = c = 0$ and $\mathbf{A}$ has to be the origin.

7 Derivatives of a function

On derivatives, we obtain new concepts, from the division by zero. We will consider the fundamentals, first.

From the viewpoint of the division by zero, when there exists the limit, at x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \infty \tag{7.1}$$

or

$$f'(x) = -\infty, \tag{7.2}$$

both cases, we can write them as follows:

$$f'(x) = 0. \tag{7.3}$$

This property was derived from the fact that the gradient of the y axis is zero; that is,

$$\tan \frac{\pi}{2} = 0, \tag{7.4}$$

that was derived from many geometric properties in [27], and also in the formal way from the result $1/0 = 0$. Of course, by the division by zero calculus, we can derive the result.

For the double angle formula

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}, \tag{7.5}$$

for $\alpha = \pi/2$, we have:

$$0 = \frac{2 \cdot 0}{1 - 0}. \tag{7.6}$$

We will look this fundamental result by elementary functions. For the function

$$y = \sqrt{1 - x^2}, \tag{7.7}$$

$$y' = \frac{-x}{\sqrt{1 - x^2}}, \tag{7.8}$$

and so,

$$[y']_{x=1} = 0, \quad [y']_{x=-1} = 0. \tag{7.9}$$

Of course, depending on the context, we should refer to the derivatives of a function at a point from the right hand direction and the left hand direction.

Here, note that, for $x = \cos\theta, y = \sin\theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1} = -\cot\theta.$$

Note also that from the expansion

$$\cot z = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty}\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right) \tag{7.10}$$

or the Laurent expansion

$$\cot z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

we have

$$\cot 0 = 0.$$

Note that in (7.10), since

$$\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right)_{\nu=0} = \frac{1}{z}, \tag{7.11}$$

we can write it simply

$$\cot z = \sum_{\nu=-\infty}^{+\infty}\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right). \tag{7.12}$$

The differential equation

$$y' = -\frac{x}{y} \tag{7.13}$$

with a general solution

$$x^2 + y^2 = a^2 \tag{7.14}$$

is satisfied for all the points of the solutions by the division by zero, however, the differential equations

$$x + yy' = 0, \quad y' \cdot \frac{y}{x} = -1 \tag{7.15}$$

are not satisfied for the all points of the solutions.

For the function $y = \log x$,

$$y' = \frac{1}{x}, \tag{7.16}$$

and so,

$$[y']_{x=0} = 0. \tag{7.17}$$

For the elementary ordinary differential equation

$$y' = \frac{dy}{dx} = \frac{1}{x}, \quad x > 0, \tag{7.18}$$

how will be the case at the point $x = 0$? From its general solution, with a general constant C

$$y = \log x + C, \tag{7.19}$$

we see that

$$y'(0) = \left[\frac{1}{x}\right]_{x=0} = 0, \tag{7.20}$$

that will mean that the division by zero $1/0 = 0$ is very natural.

In addition, note that the function $y = \log x$ has infinite order derivatives and all the values are zero at the origin, in the sense of the division by zero.

However, for the derivative of the function $y = \log x$, we have to fix the sense at the origin, clearly, because the function is not differentiable, but it has a singularity at the origin. For $x > 0$, there is no problem for (7.16) and (7.17). At $x = 0$, we see that we can not consider the limit in the sense (7.1). However, $x > 0$ we have (7.17) and

$$\lim_{x \to +0} (\log x)' = +\infty. \tag{7.21}$$

In the usual sense, the limit is $+\infty$, but in the present case, in the sense of the division by zero, we have:

$$\left[(\log x)'\right]_{x=0} = 0 \tag{7.22}$$

and we will be able to understand its sense graphically.

By the new interpretation for the derivative, we can arrange the formulas for derivatives, by the division by zero. The formula

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} \tag{7.23}$$

is very fundamental. Here, we assume that for a local one to one correspondence of the function $y = f(x)$ and for nonvanishing of the denominator

$$\frac{dy}{dx} \neq 0. \tag{7.24}$$

However, if a local one to one correspondence of the function $y = f(x)$ is ensured like the function $y = x^3$ around the origin, we do not need the assumption (7.24). Then, for the point $dy/dx = 0$, we have, by the division by zero,

$$\frac{dx}{dy} = 0. \tag{7.25}$$

This will mean that the function $x = g(y)$ has the zero derivative and practically the tangential line at the point is a parallel line to the y- axis. In this sense the formula (7.23) is valid, even the case $dy/dx = 0$. The nonvanising case, of course, the identity

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \tag{7.26}$$

holds. When we put the vanishing case, here, we obtain the identity

$$0 \times 0 = 1, \tag{7.27}$$

in a sense. Of course, it is not valid, because (7.26) is unclear for the vanishing case. Such an interesting property was referred by M. Yamane in ([22]).

In addition, for higher-order derivatives, we note the following: For a function $y = f(x) \in C^3$ whose inverse function $x = g(x)$ is single-valued, we note the formulas:

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3} \tag{7.28}$$

and

$$\frac{d^3x}{dy^3} = -\left[\frac{d^3y}{dx^3}\frac{dy}{dx} - 3\left(\frac{d^2y}{dx^2}\right)^2\right]\left(\frac{dy}{dx}\right)^{-5} \tag{7.29}$$

are valid, even at a point x_0 such that

$$f(x_0) = y_0, f'(x_0) = 0 \tag{7.30}$$

as

$$\frac{d^2x}{dy^2}(y_0) = \frac{d^3x}{dy^3}(y_0) = 0. \tag{7.31}$$

Furthermore, the formulas

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}, \tag{7.32}$$

$$\left(\frac{1}{f}\right)'' = \frac{2(f')^2 - ff''}{f^3}, \tag{7.33}$$

$$\left(\frac{1}{f}\right)''' = \frac{6ff'f'' - 6(f')^3 - f^2f'''}{f^4}, \tag{7.34}$$

..., and so on, are valid, even the case

$$f(x_0) = 0, \tag{7.35}$$

at the point x_0.

8 Differential equations

From the viewpoint of the division by zero calculus, we will see many incompleteness mathematics, in particular, in the theory of differential equations in an undergraduate level; indeed, we have considered our mathematics around an isolated singular point for analytic functions, however, we did not consider mathematics at the singular point itself. At the isolated singular point, we considered our mathematics with the limiting concept, however, the limiting values to the singular point and the value at the singular point of the function are different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. From this viewpoint, we will be able to consider differential equations even at singular points. We find many incomplete statements and problems in many undergraduate textbooks. In this section, we will point out the problems in concrete ways by examples.

This section is an extension of the paper [3].

8.1 Missing a solution

For the differential equation

$$2xydx - (x^2 - y^2)dy = 0,$$

we have a general solution with a constant C

$$x^2 + y^2 = 2Cy.$$

However, we are missing the solution $x = 0$. By this expression

$$\frac{x^2 + y^2}{C} = 2y,$$

for $C = 0$, by the division by zero, we have the missing solution $x = 0$.

For the differential equation

$$x(y')^2 - 2yy' - x = 0,$$

we have the general solution

$$C^2x^2 - 2Cy - 1 = 0.$$

However, $x = 0$ is also a solution, because

$$xdy^2 - 2ydydx - xdy^2 = 0.$$

From

$$x^2 - \frac{2y}{C} - \frac{1}{C^2} = 0,$$

by the division by zero, we obtain the solution.

For the differential equation

$$2y = xy' - \frac{x}{y'},$$

we have the general solution

$$2y = Cx^2 - \frac{1}{C}.$$

For $C = 0$, we have the solution $y = 0$, by the division by zero.

For the differential equation

$$(x^2 - a^2)(y')^2 - 2xyy' - x^2 = 0,$$

we have the general solution

$$y = Cx^2 - \left(a^2C + \frac{1}{4C}\right).$$

For $C = 0$, then $y = 0$, however, this is not a solution. But, this is the solution of the differential equation

$$(x^2 - a^2)\frac{(y')^2}{y} - 2xy' - \frac{x^2}{y} = 0.$$

For the differential equation

$$ydx + (x^2y^3 + x)dy = 0,$$

we have the general solution

$$-\frac{1}{xy} + \frac{y^2}{2} = C.$$

Of course, we have the solution $y = 0$.

For the differential equation

$$(3x^2 - 1)dy - 3xydx = 0,$$

we have the general solution

$$3x^2 + 2 = Cy^2 + 3.$$

From

$$\frac{3x^2 + 2}{C} = y^2 + \frac{3}{C},$$

we have the solution $y = 0$, by the division by zero.

8.2 Differential equations with singularities

For the differential equation

$$y' = -\frac{y}{x},$$

we have the general solution

$$y = \frac{C}{x}.$$

From the expression

$$xdy + ydx = 0,$$

we have also the general solution

$$x = \frac{C}{y}.$$

Therefore, there is no problem for the origin. Of course, $x = 0$ and $y = 0$ are the solutions.

For the differential equation

$$y' = \frac{2x - y}{x - y}, \tag{8.1}$$

we have the beautiful general solution with constant C

$$2x^2 - 2xy + y^2 = C. \tag{8.2}$$

By the division by zero calculus we see that on the whole points on the solutions (8.2) the differential equation (8.1) is satisfied. If we do not consider the division by zero, for $y = x(\neq 0)$, we will have a serious problem. However, for $x = y \neq 0$, we should consider that $y' = 0$, not by the division by zero calculus, but by $1/0 = 0$.

For the differential equation

$$y' = \frac{2xy}{x^2 - y^2}$$

and for the general solution

$$x^2 + (y - C)^2 = C^2,$$

there is no problem at the singular points $(0, 0)$ and $x = C$, Note that

$$y' = -\frac{x}{x - C}.$$

For the differential equation

$$x^3 y' = x^4 - x^2 y + 2y^2, \tag{8.3}$$

we have the general solution with constant C

$$y = \frac{x^2(x + C)}{2x + C}. \tag{8.4}$$

Note that we have also a solution $x = 0$, because,

$$x^3 dy = (x^4 - x^2 y + 2y^2)dx. \tag{8.5}$$

In particular, note that at $(0, 0)$

$$y'(0) = \frac{0}{0}, \tag{8.6}$$

and the general solution (8.4) have the value

$$y\left(-\frac{1}{2}C\right) = -\frac{1}{8}C^2, \tag{8.7}$$

by the division by zero calculus. For C tending to ∞ in the general solution, we have the another solution $y = x^2$. Then, if we understand $C = 0$, we see that the property of the solution is valid.

8.3 Continuation of solution

We will consider the differential equation

$$\frac{dx}{dt} = x^2 \cos t. \tag{8.8}$$

Then, as the general solution, we obtain, for a constant C

$$x = \frac{1}{C - \sin t}. \tag{8.9}$$

For $x_0 \neq 0$, for any given initial value (t_0, x_0) we obtain the solution satisfying the initial condition,

$$x = \frac{1}{\sin t_0 + \frac{1}{x_0} - \sin t}. \tag{8.10}$$

If

$$\left| \sin t_0 + \frac{1}{x_0} \right| < 1, \tag{8.11}$$

then the solution has many poles and L.S. Pontrjagin stated in his book that the solution is disconnected by the poles and so, the solution may be considered as infinitely many solutions.

However, by the viewpoint of the division by zero, the solution takes the value zero at the singular points and the derivatives at the singular points are all zero; that is, the solution (8.10) may be understood as one solution.

Furthermore, by the division by zero, the solution (8.10) has its sense for even the case $x_0 = 0$ and it is the solution of (8.8) satisfying the initial condition $(t_0, 0)$.

We will consider the differential equation

$$y' = y^2. \tag{8.12}$$

For $a > 0$, the solution satisfying $y(0) = a$ is given by

$$y = \frac{1}{\frac{1}{a} - x}. \tag{8.13}$$

Note that the solution satisfies on the whole space $(-\infty, +\infty)$ even at the singular point $x = \frac{1}{a}$, in the sense of the division by zero, as

$$y'\left(\frac{1}{a}\right) = y\left(\frac{1}{a}\right) = 0. \tag{8.14}$$

8.4 Singular solutions

We will consider the differential equation

$$(1 - y^2)dx = y(1 - x)dy. \tag{8.15}$$

By the standard method, we obtain the general solution, for a constant C ($C \neq 0$)

$$\frac{(x-1)^2}{C} + y^2 = 1. \tag{8.16}$$

By the division by zero, for $C = 0$, we obtain the singular solution

$$y = \pm 1.$$

For the simple Clairaut differential equation

$$y = px + \frac{1}{p}, \quad p = \frac{dy}{dx}, \tag{8.17}$$

we have the general solution

$$y = Cx + \frac{1}{C}, \tag{8.18}$$

with a general constant C and the singular solution

$$y^2 = 4x. \tag{8.19}$$

Note that we have also the solution $y = 0$ from the general solution, by the division by zero $1/0 = 0$ from $C = 0$ in (8.18).

8.5 Solutions with singularities

1). We will consider the differential equation

$$y' = \frac{y^2}{2x^2}. \tag{8.20}$$

We will consider the solution with an isolated singularity at a point a with taking the value $-2a$ in the sense of division by zero.

First, by the standard method, we have the general solution, with a constant C

$$y = \frac{2x}{1 + 2Cx}. \tag{8.21}$$

From the singularity, we have, $C = -1/2a$ and we obtain the desired solution

$$y = \frac{2ax}{a - x}. \tag{8.22}$$

Indeed, from the expansion

$$\frac{2ax}{a - x} = -2a - \frac{2a^2}{x - a}, \tag{8.23}$$

we see that it takes $-2a$ at the point a in the sense of the division by zero. This function was appeared in ([26]).

2). For any fixed $y > 0$, we will consider the differential equation

$$E(x, y)\frac{\partial E(x, y)}{\partial x} = \frac{y^2 d^2}{(y - x)^3} \tag{8.24}$$

for $0 \leq x \leq y$. Then, note that the function

$$E(x, y) = \frac{y}{y - x}\sqrt{d^2 + (y - x)^2} \tag{8.25}$$

satisfies the differential equation (8.24) satisfying the condition

$$[E(x, y)]_{x=y} = 0, \tag{8.26}$$

in the sense of the division by zero. This function was appeared in showing a strong discontinuity of the curvature center (the inversion of EM diameter) of the circle movement of the rotation of two circles with radii x and y in ([26]).

3). We will consider the singular differential equation

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} - \frac{3}{x^2}y = 0. \tag{8.27}$$

By the series expansion, we obtain the general solution, for any constants a, b

$$y = \frac{a}{x^3} + bx. \tag{8.28}$$

We see that by the division by zero

$$y(0) = 0, y'(0) = b, y''(0) = 0. \tag{8.29}$$

The solution (8.28) has its sense and the equation (8.27) is satisfied even at the origin. The value $y'(0) = b$ may be given arbitrary, however, in order to determine the value a, we have to give some value for the regular point $x \neq 0$. Of course, we can give the information at the singular point with the Laurent coefficient a, that may be interpreted with the value at the singular point zero, with the division by zero. Indeed, the value a may be considered at the value

$$[y(x)x^3]_{x=0} = a. \tag{8.30}$$

4). Next, we will consider the Euler differential equation

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0. \tag{8.31}$$

We obtain the general solution, for any constants a, b

$$y = \frac{a}{x} + \frac{b}{x^2}. \tag{8.32}$$

The solution (8.32) is satisfied even at the origin, by the division by zero and furthermore, all the derivatives of the solution of any order are all zero at the origin.

5). We will note that with the general solution, with constants C_{-2}, C_{-1}, C_0

$$y = \frac{C_{-2}}{x^2} + \frac{C_{-1}}{x} + C_0, \tag{8.33}$$

we obtain the nonlinear ordinary differential equation

$$x^2y''' + 6xy'' + 6y' = 0. \tag{8.34}$$

6). For the differential equation

$$y' = y^2(2x - 3), \tag{8.35}$$

we have the special solution

$$y = \frac{1}{(x-1)(2-x)} \tag{8.36}$$

on the interval $(1,2)$ with the singularities at $x = 1$ and $x = 2$. Since the general solution is given by, for a constant C,

$$y = \frac{1}{-x^2 + 3x + C}, \tag{8.37}$$

we can consider some conditions that determine the special solution (8.37).

8.6 Solutions with an analytic parameter

For example, in the ordinary differential equation

$$y'' + 4y' + 3y = 5e^{-3x}, \tag{8.38}$$

in order to look for a special solution, by setting $y = Ae^{kx}$ we have, from

$$y'' + 4y' + 3y = 5e^{kx}, \tag{8.39}$$

$$y = \frac{5e^{kx}}{k^2 + 4k + 3}. \tag{8.40}$$

For $k = -3$, by the division by zero calculus, we obtain

$$y = e^{-3x}\left(-\frac{5}{2}x - \frac{5}{4}\right), \tag{8.41}$$

and so, we can obtain the special solution

$$y = -\frac{5}{2}xe^{-3x}. \tag{8.42}$$

For example, for the differential equation

$$y'' + a^2 y = b\cos\lambda x, \tag{8.43}$$

we have a special solution

$$y = \frac{b}{a^2 - \lambda^2}\cos\lambda x. \tag{8.44}$$

Then, when for $\lambda = a$ (reasonance case), by the division by zero calculus, we obtain the special solution

$$y = \frac{bx\sin(ax)}{2a} + \frac{b\cos(ax)}{4a^2}. \tag{8.45}$$

Recall the Newton kernel, for $N > 2$,

$$\Gamma_N(x,y) = \frac{1}{N(2-N)\omega_N}|x-y|^{2-N} \tag{8.46}$$

and

$$\Gamma_2(x,y) = \frac{1}{2\pi}\log|x-y|, \tag{8.47}$$

where

$$\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}.$$

From $\Gamma_N(x,y)$, by the division by zero calculus, we have:

$$\frac{1}{2\pi}\log|x-y| + \frac{1}{4\pi}(\gamma + \log\pi), \tag{8.48}$$

where γ is the Euler constant.

For the Green function $G_N(x,y)$ of the Laplace operator on the ball with center a and radius r on the Euclidean space of $N (N \geq 3)$ dimension is given by

$$G_N(x,y) = \|x-y\|^{2-N} - \left(\frac{r}{\|y-a\|}\frac{1}{\|x-y^*\|}\right)^{N-2}, \tag{8.49}$$

where y^* is the inversion of y

$$y^* - a = \left(\frac{r}{\|y-a\|}\right)^2 (y-a). \tag{8.50}$$

By $N = 2$, we obtain the corresponding formula, by the division by zero calculus,

$$G_2(x,y) = \log\left(\frac{\|y-a\|}{r}\frac{\|x-y^*\|}{\|x-y\|}\right) \tag{8.51}$$

([4], page 91).

We can find many examples.

8.7 Special reductions by division by zero of solutions

For the differential equation

$$y'' - (a+b)y' + aby = e^{cx}, c \neq a, b; a \neq b,$$

we have the special solution

$$y = \frac{e^{cx}}{(c-a)(c-b)}.$$

If $c = a(\neq b)$, then, by the division by zero calculus, we have

$$y = \frac{xe^{cx}}{a-b}.$$

If $c = a = b$, then, by the division by zero calculus, we have

$$y = \frac{x^2 e^{cx}}{2}.$$

For the differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0,$$

we obtain the general solution, for $\gamma^2 > 4mk$

$$x(t) = e^{-\alpha t}\left(C_1 e^{\beta t} + C_2 e^{-\beta t}\right)$$

with

$$\alpha = \frac{\gamma}{2m}$$

and

$$\beta = \frac{1}{2m}\sqrt{\gamma^2 - 4mk}.$$

For $m = 0$, by the division by zero calculus we obatin the reasonable solution $\alpha = 0$ and $\gamma = -k/\gamma$.

We will consider the differential equation, for a constant K

$$y' = Ry.$$

Then, we have the general solution

$$y(x) = y(0)e^{Rt}.$$

For the differential equation

$$y' = Ry\left(1 - \frac{y}{K}\right),$$

we have the solution

$$y = \frac{y(0)e^{Rt}}{1 + \frac{y(0)(e^{Rt}-1)}{K}}.$$

If $K = 0$, then, by the division by zero, we obtain the previous result, immediately.

We will consider the fundamental ordinary differential equations

$$x''(t) = g - kx'(t) \tag{8.52}$$

with the initial conditions

$$x(0) = -h, x'(0) = 0. \tag{8.53}$$

Then we have the solution

$$x(t) = \frac{g}{k}t + \frac{g(e^{-kt} - 1)}{k^2} - h. \tag{8.54}$$

Then, for $k = 0$, we obtain, immediately, by the division by zero calculus

$$x(t) = \frac{1}{2}gt^2 - h. \tag{8.55}$$

For the differential equation

$$x''(t) = g - k(x'(t))^2 \tag{8.56}$$

satisfying the same condition with (8.54), we obtain the solution

$$x(t) = \frac{1}{2k}\log\frac{\left(e^{2t\sqrt{kg}} + 1\right)^2}{4e^{2t\sqrt{kg}}} - h. \tag{8.57}$$

Then, for $k = 0$, we obtain

$$x(t) = \frac{1}{2}gt^2 - h. \tag{8.58}$$

immediately, by the division by zero calculus.

For the differential equation

$$mx''(t) = -mg - rx'(t),$$

the solution satisfying the conditions $x(0) = x_0, x'(0) = v_0$ is given by

$$x(t) = -\frac{g}{r}mt + A + B\exp\left(-\frac{r}{m}t\right),$$

with

$$A = x_0 - B, B = -\frac{m}{r}\left(\frac{m}{r}g + v_0\right).$$

For $r = 0$, by the division by zero calculus, we have the reasonable solution

$$x(t) = -\frac{1}{2}gt + v_0t + x_0.$$

For the differential equation

$$x''(t) = -g + k(x'(t))^2 \tag{8.59}$$

satisfying the initial conditions

$$x(0) = 0, x'(0) = V, \tag{8.60}$$

we have

$$x'(t) = -\sqrt{\frac{g}{k}}\tan(\sqrt{kg}t - \alpha), \tag{8.61}$$

with

$$\alpha = \tan^{-1}\sqrt{\frac{k}{g}}V \tag{8.62}$$

and the solution

$$x(t) = \frac{1}{k}\log\frac{\cos\left(\sqrt{kg}t - \alpha\right)}{\cos\alpha}. \tag{8.63}$$

Then we obtain for $k = 0$, by the division by zero calculus

$$x'(t) = -gt + V \tag{8.64}$$

and

$$x(t) = -\frac{1}{2}gt^2 + Vt. \tag{8.65}$$

We will consider the typical ordinary differential equation

$$mx''(t) = mg - m(\lambda x'(t) + \mu(x'(t))^2), \tag{8.66}$$

satisfying the initial conditions

$$x(0) = x'(0) = 0. \tag{8.67}$$

Then we have the solution

$$x(t) = \frac{-\lambda + \sqrt{\lambda^2 + 4\mu g}}{2\mu} t + \frac{1}{\mu} \log[(\frac{-\lambda + \sqrt{\lambda^2 + 4\mu g}}{2\mu} \exp(-\sqrt{\lambda^2 + 4\mu g} t) + \frac{\lambda + \sqrt{\lambda^2 + 4\mu g}}{2\mu}) \frac{\mu}{\sqrt{\lambda^2 + 4\mu g}}]. \tag{8.68}$$

Then, if $\mu = 0$, we obtain, immediately, by the division by zero calculus

$$x(t) = \frac{g}{\lambda} t + \frac{1}{\lambda^2} g e^{-\lambda t} - \frac{g}{\lambda^2}. \tag{8.69}$$

Furthermore, if $\lambda = 0$, then we have

$$x(t) = \frac{1}{2} g t^2. \tag{8.70}$$

We can find many and many such examples. However, note that the following fact.

For the differential equation

$$y''' + a^2 y' = 0, \tag{8.71}$$

we obtain the general solution, for $a \neq 0$

$$y = A \sin ax + B \cos ax + C. \tag{8.72}$$

For $a = 0$, from this general solution, how can we obtain the correspondent solution

$$y = Ax^2 + Bx + C, \tag{8.73}$$

naturally?

For the differential equation

$$y' = ae^{\lambda x}y^2 + afe^{\lambda x}y + \lambda f, \tag{8.74}$$

we obtain a special solution, for $a \neq 0$

$$y = -\frac{\lambda}{a}e^{-\lambda x}. \tag{8.75}$$

For $a = 0$, from this solution, how can we obtain the correspondent solution

$$y = \lambda fx + C, \tag{8.76}$$

naturally?

8.8 Partial differential equations

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bx\frac{\partial w}{\partial x} + (cx + d)w, \tag{8.77}$$

we have a special solution

$$w(x,t) = \exp\left[-\frac{c}{b}x + \left(d + \frac{ac^2}{b^2}\right)t\right]. \tag{8.78}$$

For $b = 0$, how will be the correspondent solution? If $b = 0$, then $c = 0$ and

$$\frac{c}{b} = \frac{0}{0} = 0, \tag{8.79}$$

and we obtain the correspondent solution.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (be^{\beta t} + c)w, \tag{8.80}$$

we have special solutions

$$w(x,t) = (Ax + B)\exp\left[\frac{b}{\beta}e^{\beta t} + ct\right], \tag{8.81}$$

$$w(x,t) = A(x^2 + 2at) \exp\left[\frac{b}{\beta} e^{\beta t} + ct\right], \tag{8.82}$$

and

$$w(x,t) = A \exp\left[\lambda x + a\lambda^2 t + \frac{b}{\beta} e^{\beta t} + ct\right]. \tag{8.83}$$

Then, we see that for $\beta = 0$, by the interpretation

$$\left[\frac{1}{\beta} e^{\beta t}\right]_{\beta=0} = t, \tag{8.84}$$

we can obtain the correspondent solutions.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (bxe^{\beta x} + c)w, \tag{8.85}$$

we have a special solution

$$w(x,t) = A \exp\left[\frac{b}{\beta} x e^{\beta t} + \frac{ab^2}{2\beta^3} e^{2\beta t} + ct\right]. \tag{8.86}$$

Then, for $\beta = 0$, by the interpretation

$$\left[\frac{1}{\beta^j} e^{\beta t}\right]_{\beta=0} = \frac{1}{j!} t^j, \tag{8.87}$$

we can obtain the correspondent solution.

However, the above properties will be, in general, complicated.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + bw, \tag{8.88}$$

we have the fundamental solution

$$w(x,t) = \frac{1}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at} + bt\right). \tag{8.89}$$

For $a = 0$, we have the correspondent solution

$$w(x,t) = \exp bt. \tag{8.90}$$

For the factor

$$\frac{1}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at}\right) \tag{8.91}$$

we have, for letting $a \to 0$,

$$\delta(x), \tag{8.92}$$

meanwhile, at $a = 0$, by the division by zero calculus, we have 0. So, the reduction problem is a delicate open problem.

For the partial differential equation

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + (-bx^2 + ct + d)w, \tag{8.93}$$

we have a special solution

$$w(x,t) = \exp\left[\frac{1}{2}\sqrt{\frac{b}{a}}x^2 + \frac{1}{2}ct^2 + (\sqrt{ab} + d)t\right]. \tag{8.94}$$

For $a = 0$, how will be the correspondent solution? Since we have the solution

$$w(x,t) = \exp\left[-bx^2 t + \frac{1}{2}ct^2 + dt\right], \tag{8.95}$$

for the factor

$$\frac{1}{2}\sqrt{\frac{b}{a}}x^2 \tag{8.96}$$

we have to have

$$-bx^2 t. \tag{8.97}$$

8.9 Open problems

As the important open problems, we would like to propose them clearly.

We have considered our mathematics around an isolated singular point for analytic functions, however, we did not consider mathematics at the singular point itself. At the isolated singular point, we consider our mathematics with the limiting concept, however, the limiting values to the singular point and on the values at the singular point in the sense of division by zero calculus are different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. We thus have a general open problem discussing our mathematics on a domain containing the singular point.

We refer to the reduction problems by concrete examples; there we found the delicate property. For this interesting property we expect some general theory.

9 Euclidean spaces and division by zero

In this section, we will see the division by zero properties on the Euclidean spaces. Since the impact of the division by zero and division by zero calculus is widely expanded in elementary mathematics, here, elementary topics will be introduced as the first stage.

9.1 Broken phenomena of figures by area and volume

The strong discontinuity of the division by zero around the point at infinity will be appeared as the broken of various figures. These phenomena may be looked in many situations as the universe one. However, the simplest cases are disc and sphere (ball) with radius $1/R$. When $R \to +0$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $R = 0$, they are zero, because they become the half-plane and half-space, respectively. These facts may be also looked by analytic geometry, as we see later. However, the results are clear already from the definition of the division by zero:

For this fact, note the following:

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$ (see [2]). We thus see that

$$\lim_{z\to\infty} z = \infty, \tag{9.1}$$

however,

$$[z]_{z=\infty} = 0, \tag{9.2}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (9.1) and (9.2) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x\to+\infty} x = +\infty, \quad \lim_{x\to-\infty} x = -\infty, \tag{9.3}$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0. \tag{9.4}$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [27], we gave many examples for this property.

In particular, in $z \to \infty$ in (9.1), ∞ represents the topological point on the Riemann sphere, meanwhile ∞ in the left hand side in (9.1) represents the limit by means of the ϵ - δ logic.

9.2 Parallel lines

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2. \tag{9.5}$$

The common point is given by, if $a_1 b_2 - a_2 b_1 \neq 0$; that is, the lines are not parallel

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \right). \tag{9.6}$$

By the division by zero, we can understand that if $a_1 b_2 - a_2 b_1 = 0$, then the commom point is always given by

$$(0, 0), \tag{9.7}$$

even the two lines are the same. This fact shows that the image of the Euclidean space in Section 3 is right.

In particular, note that the concept of parallel lines is very important in the Euclidean plane and non-Euclidean geometry. In our sense, there is no parallel line and all lines pass the origin. This will be our world in the Euclidean plane. However, this property is not geometrical and has a strong discontinuity. This surprising property may be looked clearly by the polar representation of a line.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)}, \tag{9.8}$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle OP ($P = (r, \theta)$ on the line) and the positive x axis. Then, if $\theta - \alpha = \pi/2$: that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

This will mean simply that any line arrives at the point at infinity and the point is represented by zero and so, for the line we can add the point at the origin. In this sense, we can add the origin to any line as the point of the compactification of the line. This surprising new property may be looked in our mathematics grobally.

The distance d from the origin to the line determined by the two planes

$$\Pi_k : a_k x + b_k y + c_k z = 1, k = 1, 2, \tag{9.9}$$

is given by

$$d = \sqrt{\frac{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}. \tag{9.10}$$

If the two lines are coincident, then, of course, $d = 0$. However, if the two planes are parallel, by the division by zero, $d = 0$. This will mean that any plane contains the origin as in a line.

9.3 Tangential lines and $\tan \frac{\pi}{2} = 0$

We looked the very fundamental and important formula $\tan \frac{\pi}{2} = 0$ in Section 6. In this subsection, for its importance we will furthermore see its various geometrical meanings.

We consider the high $\tan \theta \left(0 \leq \theta \leq \frac{\pi}{2}\right)$ that is given by the common point of two lines $y = (\tan \theta)x$ and $x = 1$ on the (x, y) plane. Then,

$$\tan \theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$\tan \frac{\pi}{2} = 0,$$

by the division by zero. The result will show that, when $\theta = \pi/2$, two lines $y = (\tan \theta)x$ and $x = 1$ do not have a common point, because they are parallel in the usual sense. However, in the sense of the division by zero, parallel lines have the common point $(0, 0)$. Therefore, we can see the result $\tan \frac{\pi}{2} = 0$ following our new space idea.

We consider general lines represented by

$$ax + by + c = 0, a'x + b'y + c' = 0. \tag{9.11}$$

The gradients are given by

$$k = -\frac{a}{b}, k' = -\frac{a'}{b'}, \tag{9.12}$$

respectively. In particular, note that if $b = 0$, then $k = 0$, by the division by zero.

If $kk' = -1$, then the lines are orthogonal; that is,

$$\tan\frac{\pi}{2} = 0 = \pm\frac{k - k'}{1 + kk'}, \tag{9.13}$$

which shows that the division by zero $1/0 = 0$ and orthogonality meets in a very good way.

Furthermore, even in the case of polar coordinates $x = r\cos\theta, y = r\sin\theta$, we can see the division by zero

$$\tan\frac{\pi}{2} = \frac{y}{0} = 0. \tag{9.14}$$

In particular, note that:

From the expansion

$$\tan z = -\sum_{\nu=-\infty}^{+\infty}\left(\frac{1}{z - (2\nu - 1)\pi/2} + \frac{1}{(2\nu - 1)\pi/2}\right), \tag{9.15}$$

$$\tan\frac{\pi}{2} = 0.$$

The division by zero may be looked even in the rotation of the coordinates.

We will consider a 2 dimensional curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{9.16}$$

and a rotation defined by

$$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta. \tag{9.17}$$

Then, we write, by inserting these (x, y)

$$AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0. \tag{9.18}$$

Then,

$$H = 0 \Longleftrightarrow \tan 2\theta = \frac{2h}{a-b}. \tag{9.19}$$

If $a = b$, then, by the division by zero,

$$\tan\frac{\pi}{2} = 0, \quad \theta = \frac{\pi}{4}. \tag{9.20}$$

For $h^2 > ab$, the equation

$$ax^2 + 2hxy + by^2 = 0 \tag{9.21}$$

represents 2 lines and the angle θ made by two lines is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}. \tag{9.22}$$

If $h^2 - ab = 0$, then, of course, $\theta = 0$. If $a + b = 0$, then, by the division by zero, $\theta = \pi/2$ from $\tan\theta = 0$.

For a hyperbolic function

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad a, b > 0 \tag{9.23}$$

the angle θ maden by the two asymptotic lines $y = \pm(b/a)x$ is given by

$$\tan\theta = \frac{2(b/a)}{1-(b/a)^2}. \tag{9.24}$$

If $a = b$, then $\theta = \pi/2$ from $\tan\theta = 0$.

For a line

$$x\cos\theta + y\sin\theta - p = 0 \tag{9.25}$$

and for data (x_j, y_j), the minimum of $\sum_{j=1}^n d_j^2$ for the distance d_j of the plane and the point (x_j, y_j) is attained for the case

$$\tan 2\theta = \frac{2\gamma_x y\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}, \tag{9.26}$$

where

$$\gamma_{xy} = \frac{n \sum_j x_j y_j - (\sum_j x_j)(\sum_j y_j)}{n^2 \sigma_x \sigma_y}$$

and

$$\sigma_x = \frac{1}{n}\sqrt{n \sum_j x_j^2 - (\sum_j x_j)^2}.$$

If $\sigma_x^2 = \sigma_y^2$, then $\theta = \pi/4$ from $\tan 2\theta = 0$.

We consider the unit circle with center at the origin on the (x, y) plane. We consider the tangential line for the unit circle at the point that is the common point of the unit circle and the line $y = (\tan\theta)x$ $\left(0 \le \theta \le \frac{\pi}{2}\right)$. Then, the distance R_θ between the common point and the common point of the tangential line and x-axis is given by

$$R_\theta = \tan\theta.$$

Then,

$$R_0 = \tan 0 = 0,$$

and

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$R_{\pi/2} = \tan\frac{\pi}{2} = 0.$$

This example shows also that by the stereoprojection mapping of the unit sphere with center the origin $(0, 0, 0)$ onto the plane, the north pole corresponds to the origin $(0, 0)$.

In this case, we consider the orthogonal circle C_{R_θ} with the unit circle through at the common point and the symmetric point with respect to the x-axis with center $((\cos\theta)^{-1}, 0)$. Then, the circle C_{R_θ} is as follows:

C_{R_0} is the point $(1, 0)$ with curvature zero, and $C_{R_{\pi/2}}$ (that is, when $R_\theta = \infty$, in the common sense) is the y-axis and its curvature is also zero. Meanwhile, by the division by zero, for $\theta = \pi/2$ we have the same result, because $(\cos(\pi/2))^{-1} = 0$.

Note that from the expansion

$$\frac{1}{\cos z} = 1 + \sum_{\nu=-\infty}^{+\infty} (-1)^\nu \left(\frac{1}{z - (2\nu - 1)\pi/2} + \frac{2}{(2\nu - 1)\pi} \right), \tag{9.27}$$

$$\left(\frac{1}{\cos z}\right)\left(\frac{\pi}{2}\right) = 1 - \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1} = 0.$$

The point $(\cos\theta, 0)$ and $((\cos\theta)^{-1}, 0)$ are the symmetric points with respective to the unit circle, and the origin corresponds to the origin.

In particular, the formal calculation

$$\sqrt{1 + R_{\pi/2}^2} = 1 \tag{9.28}$$

is not good. The identity $\cos^2\theta + \sin^2\theta = 1$ is valid always, however $1 + \tan^2\theta = (\cos\theta)^{-2}$ is not valid for $\theta = \pi/2$.

Note that from the expansin

$$\frac{1}{\cos^2 z} = \sum_{\nu=-\infty}^{+\infty}\frac{1}{(z-(2\nu-1)\pi/2)^2}, \tag{9.29}$$

$$\left(\frac{1}{\cos^2 z}\right)\left(\frac{\pi}{2}\right) = \frac{2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{1}{3}.$$

On the point $(p, q)(0 \le p, q \le 1)$ on the unit circle, we consider the tangential line $L_{p,q}$ of the unit circle. Then, the common points of the line $L_{p,q}$ with x-axis and y-axis are $(1/p, 0)$ and $(0, 1/q)$, respectively. Then, the area S_p of the triangle formed by the three points $(0,0)$, $(1/p, 0)$ and $(0, 1/q)$ is given by

$$S_p = \frac{1}{2pq}.$$

Then,

$$p \longrightarrow 0; \quad S_p \longrightarrow +\infty,$$

however,

$$S_0 = 0$$

(H. Michiwaki: 2015.12.5).

We denote the point on the unit circle on the (x, y) with $(\cos\theta, \sin\theta)$ for the angle θ with the positive real line. Then, the tangential line of the unit circle at the point meets at the point $(R_\theta, 0)$ for $R_\theta = [\cos\theta]^{-1}$ with the x-axis for the case $\theta \neq \pi/2$. Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty, \tag{9.30}$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty, \tag{9.31}$$

however,

$$R_{\pi/2} = \left[\cos\left(\frac{\pi}{2}\right)\right]^{-1} = 0, \tag{9.32}$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$ (H. Michiwaki: 2015.12.5).

The line through the points $(0, 1)$ and $(\cos\theta, \sin\theta)$ meets the x axis with the point $(R_\theta, 0)$ for the case $\theta \neq \pi/2$ by

$$R_\theta = \frac{\cos\theta}{1 - \sin\theta}. \tag{9.33}$$

Then,

$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty, \tag{9.34}$$

$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty, \tag{9.35}$$

however,

$$R_{\pi/2} = 0, \tag{9.36}$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$.

Note also that

$$\left[1 - \sin\left(\frac{\pi}{2}\right)\right]^{-1} = 0.$$

For a smooth curve $C : r = r(\theta), r \neq 0$, we consider the tangential line at P and a near point Q on the curve C. Let H be the nearest point on the line OP, O is the pole of the coordinate and $\delta\theta$ is the angle for the line OP to the line OG. Then, we have

$$\tan\Theta := \lim_{\delta\theta\to 0} \frac{QH}{PH} = \frac{r(\theta)}{r'(\theta)}. \tag{9.37}$$

If $r'(\theta_0) = 0$, then $\tan\Theta = 0$ and $\Theta = \pi/2$, and the result is reasonable.

For the parabolic equation $y^2 = 4ax, a > 0$, at a point (x, y), the normal line shadow on the x-axis is given by

$$|yy'| = 2a. \tag{9.38}$$

At the origin, we have, from $y'(0) = 0$,

$$|yy'| = 0. \tag{9.39}$$

For the equation

$$x^m y^n = a^{m+n}, \quad a, m, n > 0, \tag{9.40}$$

let P be a point (x, y) on the curve. Let $T(x.x + (n/m)x)$ be the x cut of the tagential line of the curve and put $M(x, 0)$. Then, we have

$$TM : OM = -\frac{n}{m}. \tag{9.41}$$

This formula is valid for the cases $n = 0$ and $m = 0$, by the division by zero. Note that for the both lines $x = a$ and $y = a$, $y' = 0$.

9.4 Two Circles

We consider two circles with radii a, b with centers $(a, 0); a > 0$ and $(-b, 0); b > 0$, respectively. Then, the external common tangents $L_{a,b}$ (we assume that $a < b$ and that $L_{a,b}$ is not the y axis) has the common point with the x-axis at $(R_a, 0)$ which is given by, by fixing b

$$R_a = \frac{2ab}{b-a}. \tag{9.42}$$

We consider the circle C_{R_a} with center at $(R_a, 0)$ with radius R_a. Then,

$$a \to b \Longrightarrow R_a \to \infty,$$

however, when $a = b$, then we have $R_b = -2b$ by the division by zero, from the identity

$$\frac{2ab}{b-a} = -2b - \frac{2b^2}{a-b}.$$

Meanwhile, when we interpret (9.38) as

$$R_a = \frac{-1}{a-b} \cdot 2ab, \tag{9.43}$$

we have, for $a = b$, $R_b = 0$. It means that the circle C_{R_b} is the y axis with curvature zero through the origin $(0, 0)$.

The above formulas will show strong discontinuity for the change of the a and b from $a = b$ (H. Okumura: 2015.10.29).

We denote the circles S_j:

$$(x - a_j)^2 + (y - b_j)^2 = r_j^2. \tag{9.44}$$

Then, the common point (X, Y) of the co- and exterior tangential lines of the circles S_j for $j = 1, 2$,

$$(X, Y) = \left(\frac{r_1 a_2 - r_2 a_1}{r_1 - r_2}, \frac{r_1 b_2 - r_2 b_1}{r_1 - r_2} \right). \tag{9.45}$$

We will fix the circle S_2. Then, from the expansion

$$\frac{r_1 a_2 - r_2 a_1}{r_1 - r_2} = \frac{r_2(a_2 - a_1)}{r_1 - r_2} + a_2 \tag{9.46}$$

for $r_1 = r_2$, by the division by zero, we have

$$(X, Y) = (a_2, b_2). \tag{9.47}$$

Meanwhile, when we interpret (9.42) as

$$\frac{r_1 a_2 - r_2 a_1}{r_1 - r_2} = \frac{1}{r_1 - r_2} \cdot (r_1 a_2 - r_2 a_1), \tag{9.48}$$

we obtain that

$$(X, Y) = (0, 0), \tag{9.49}$$

that is reasonable. However, the both cases, the results show strong discontinuity.

9.5 Newton's method

The Newton's method is fundamental when we look for the solutions for some general equation $f(x) = 0$ numerically and practically. We will refer to its prototype case.

We will assume that a function $y = f(x)$ belongs to C^1 class. We consider the sequence $\{x_n\}$ for $n = 0, 1, 2, \ldots, n, \ldots$, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots. \tag{9.50}$$

When $f(x_n) = 0$, we have
$$x_{n+1} = x_n, \tag{9.51}$$
in the reasonable way. Even the case $f'(x_n) = 0$, we have also the reasonable result (9.47), by the division by zero.

9.6 Halley's method

As in the Newton's method, in order to look for the solution of the equation $f(x) = 0$, we consider the series
$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{a[f'(x_n)]^2 - f(x_n)f''(x_n)}$$
and
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 - \frac{f(x_n)}{f'(x_n)}\frac{f''(x_n)}{2f'(x_n)}\right]^{-1}.$$
If $f(x_n) = 0$, the processes stop and there is no problem. Even the case $f'(x_n) = 0$, the situation is similar.

9.7 Cauchy's mean value theorem

For the Cauchy mean value theorem: for $f, g \in Differ(a, b)$, differentiable, and $\in C^0[a, b]$, continuous and if $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, then there exists $\xi \in (a, b)$ satisfying that
$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(\xi)}{g'(\xi)}, \tag{9.52}$$
we do not need the assumptions $g(a) \neq g(b)$ and $f'(x)^2 + g'(x)^2 \neq 0$, by the division by zero. Indeed, if $g(a) = g(b)$, then, by the Rolle theorem, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$. Then, the both terms are zero and the equality is valid.

For $f, g \in C^2[a, b]$, there exists a $\xi \in (a, b)$ satisfying
$$\frac{f(b) - f(a) - (b - a)f'(a)}{g(b) - g(a) - (b - a)g'(a)} = \frac{f''(a)}{g''(a)}.$$
Here, we do not need the assumption
$$g(b) - g(a) - (b - a)g'(a) \neq 0,$$
by the division by zero.

9.8 Length of tangential lines

We will consider a function $y = f(x)$ of C^1 class on the real line. We consider the tangential line through $(x, f(x))$

$$Y = f'(x)(X - x) + f(x). \tag{9.53}$$

Then, the length (or distance) $d(x)$ between the point $(x, f(x))$ and $\left(x - \frac{f(x)}{f'(x)}, 0\right)$ is given by, for $f'(x) \neq 0$

$$d(x) = |f(x)|\sqrt{1 + \frac{1}{f'(x)^2}}. \tag{9.54}$$

How will be the case $f'(x^*) = 0$? Then, the division by zero shows that

$$d(x^*) = |f(x^*)|. \tag{9.55}$$

Meanwhile, the x axis point $(X_t, 0)$ of the tangential line at (x, y) and y axis point $(0, Y_n)$ of the normal line at (x, y) are given by

$$X_t = x - \frac{f(x)}{f'(x)} \tag{9.56}$$

and

$$Y_n = y + \frac{x}{f'(x)}, \tag{9.57}$$

respectively. Then, if $f'(x) = 0$, we obtain the reasonable results:

$$X_t = x, \quad Y_n = y. \tag{9.58}$$

9.9 Curvature and center of curvature

We will assume that a function $y = f(x)$ is of class C^2. Then, the curvature radius ρ and the center $O(x, y)$ of the curvature at point $(x, f(x))$ are given by

$$\rho(x, y) = \frac{(1 + (y')^2)^{3/2}}{y''} \tag{9.59}$$

and

$$O(x, y) = \left(x - \frac{1 + (y')^2}{y''}y', y + \frac{1 + (y')^2}{y''}\right), \tag{9.60}$$

respectively. Then, if $y'' = 0$, we have:

$$\rho(x, y) = 0 \tag{9.61}$$

and

$$O(x, y) = (x, y), \tag{9.62}$$

by the division by zero. They are reasonable.

We will consider a curve $\mathbf{r} = \mathbf{r}(s), s = s(t)$ of class C^2. Then,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \mathbf{t} = \frac{d\mathbf{r}(\mathbf{s})}{ds}, v = \frac{ds}{dt}, \frac{d\mathbf{t}(\mathbf{s})}{ds} = \frac{1}{\rho}\mathbf{n},$$

by the principal normal unit vector $\mathbf{n}$. Then, we see that

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{t} + \frac{v^2}{\rho}\mathbf{n}.$$

If $\rho(s_0) = 0$, then

$$\mathbf{a}(s_0) = \left[\frac{dv}{dt}\mathbf{t}\right]_{s=s_0} \tag{9.63}$$

and

$$\left[\frac{v^2}{\rho}\right]_{s=s_0} = \infty \tag{9.64}$$

will be funny. It will be the zero.

9.10 $n = 2, 1, 0$ regular polygons inscribed in a disc

We consider n regular polygons inscribed in a fixed disc with radius a. Then we note that their area S_n and the lengths L_n of the sum of the sides are given by

$$S_n = \frac{na^2}{2} \sin \frac{2\pi}{n} \tag{9.65}$$

and

$$L_n = 2na \sin \frac{\pi}{n}, \tag{9.66}$$

respectively. For $n \geq 3$, the results are clear.

For $n = 2$, we will consider two diameters that are the same. We can consider it as a generalized regular polygon inscribed in the disc as a degenerate case. Then, $S_2 = 0$ and $L_2 = 4a$, and the general formulas are valid.

Next, we will consider the case $n = 1$. Then the corresponding regular polygon is a just diameter of the disc. Then, $S_1 = 0$ and $L_1 = 0$ that will mean that any regular polygon inscribed in the disc may not be formed and so its area and length of the side are zero.

For a $n = 1$ triangle, if 1 means one side, then we can interpretate as in the above, however, if we consider 1 as one vertex, the above situation may be consider as one point on the circle which coincides with $S_l = L_l = 0$.

Now we will consider the case $n = 0$. Then, by the division by zero calculus, we obtain that $S_0 = \pi a^2$ and $L_0 = 2\pi a$. Note that they are the area and the length of the disc. How to understand the results? Imagine contrary n tending to infinity, then the corresponding regular polygons inscribed in the disc tend to the disc. Recall our new idea that the point at infinity is represented by 0. Therefore, the results say that $n = 0$ regular polygons are $n = \infty$ regular polygons inscribed in the disc in a sense and they are the disc. This is our interpretation of the theorem:

Theorem. $n = 0$ *regular polygons inscribed in a disc are the whole disc.*

In addition, note that each inner angle A_n of a general n regular polygon inscribed in a fixed disc with radius a is given by

$$A_n = \left(1 - \frac{2}{n}\right)\pi. \tag{9.67}$$

The circumstances are similar for n regular polygons circumscribed in the disc, because the corresponding data are given by

$$S_n = na^2 \tan\frac{\pi}{n} \tag{9.68}$$

and

$$L_n = 2na \tan\frac{\pi}{n}, \tag{9.69}$$

and (8.63), respectively.

9.11 Our life figure

As an interesting figure which shows an interesting relation between 0 and infinity, we will consider a sector Δ_α on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{ |\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2} \right\}.$$

We will consider a disc inscribed in the sector Δ_α whose center $(k, 0)$ with radius r. Then, we have

$$r = k \sin \alpha. \tag{9.70}$$

Then, note that as k tends to zero, r tends to zero, meanwhile k tends to $+\infty$, r tends to $+\infty$. However, by our division by zero calculus, we see that immediately that

$$[r]_{r=\infty} = 0. \tag{9.71}$$

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to the origin suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

9.12 H. Okumura's example

The suprising example by H. Okumura will show a new phenomenon at the point at infinity.

On the sector Δ_α, we shall change the angle and we consider a fixed circle $C_a, a > 0$ with radius a inscribed in the sectors. We see that when the circle tends to $+\infty$, the angles α tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}. \tag{9.72}$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive imaginary line is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

9.13 Interpretation by analytic geometry

The results in Subsection 9.1 may be interpretated beautifully by analytic geometry and matrix theory.

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2, 3. \tag{9.73}$$

The area S of the triangle surrounded by these lines is given by

$$S = \pm\frac{1}{2} \cdot \frac{\triangle^2}{D_1 D_2 D_3}, \tag{9.74}$$

where $\triangle$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and D_k is the co-factor of $\triangle$ with respect to c_k. $D_k = 0$ if and only if the corresponding lines are parallel. $\triangle = 0$ if and only if the three lines are parallel or they have a common point. We can see that the degeneracy (broken) of the triangle may be interpretated by $S \neq 0$ beautifully, by the division by zero.

For a function

$$S(x, y) = a(x^2 + y^2) + 2gx + 2fy + c, \tag{9.75}$$

the radius R of the circle $S(x, y) = 0$ is given by

$$R = \sqrt{\frac{g^2 + f^2 - ac}{a^2}}. \tag{9.76}$$

If $a = 0$, then the area πR^2 of the disc is zero, by the division by zero; that is, the circle is a line (degenerate).

The center of the circle (9.71) is given by

$$\left(-\frac{g}{a}, -\frac{f}{a}\right). \tag{9.77}$$

Therefore, the center of a general line

$$2gx + 2fy + c = 0 \tag{9.78}$$

may be considered as the origin $(0, 0)$, by the division by zero.

On the compile z plane, a circle containing a line is represented by the equation

$$az\overline{z} + \overline{\alpha}z + \alpha\overline{z} + c = 0, \tag{9.79}$$

for a, c : real and $ac \leq \overline{\alpha}\alpha$. Then the center and the radius are given by

$$-\frac{\alpha}{a} \tag{9.80}$$

and

$$\frac{\sqrt{\alpha\overline{\alpha} - ac}}{a}, \tag{9.81}$$

respectively. If $a = 0$, then it is a line with center $(0, 0)$ with radius 0, by the division by zero. The curvature of the line is, of course, zero, by the division by zero.

We consider the functions

$$S_j(x, y) = a_j(x^2 + y^2) + 2g_j x + 2f_j y + c_j. \tag{9.82}$$

The distance d of the centers of the circles $S_1(x, y) = 0$ and $S_2(x, y) = 0$ is given by

$$d^2 = \frac{g_1^2 + f_1^2}{a_1^2} - 2\frac{g_1 g_2 + f_1 f_2}{a_1 a_2} + \frac{g_2^2 + f_2^2}{a_2^2}. \tag{9.83}$$

If $a_1 = 0$, then by the division by zero

$$d^2 = \frac{g_2^2 + f_2^2}{a_2^2}. \tag{9.84}$$

Then, $S_1(x, y) = 0$ is a line and its center is the origin $(0, 0)$. Therefore, the result is very reasonable.

Meanwhile, the identity $\cos^2\theta + \sin^2\theta = 1$ is valid always, however $1 + \tan^2\theta = (\cos\theta)^{-2}$ is not valid for $\theta = \pi/2$, in the sense of the division by zero, because we consider the formula at $\theta = \pi/2$, with not the limiting values.

The distance d between two lines given by

$$\frac{x - a_j}{L_1} = \frac{y - b_j}{M_j} = \frac{z - c_j}{N_j}, \quad j = 1, 2, \tag{9.85}$$

is given by

$$d = \frac{\begin{vmatrix} a_2 - a_1 & b_2 - b_1 & c_2 - c_1 \\ L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \end{vmatrix}}{\sqrt{(M_l N_2 - M_2 N_1)^2 + (N_l L_2 - N_2 L_1)^2 + (L_l M_2 - L_2 M_1)^2}}. \tag{9.86}$$

If two lines are parallel, then we have $d = 0$.

9.14 Interpretation with volumes

We write four planes by

$$\pi_k : a_k x + b_k y + c_k z + d_k = 0, k = 1, 2, 3, 4. \tag{9.87}$$

The volume V of the tetrahedron surrounded by these planes is given by

$$V = \pm \frac{1}{6} \cdot \frac{\triangle^2}{D_1 D_2 D_3 D_4}, \tag{9.88}$$

where $\triangle$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and D_k is the co-factor of $\triangle$ with respect to d_k. $D_k = 0$ if and only if two planes of the corresponding three planes are parallel. $\triangle = 0$ if and only if the four planes π_k contain four lines L_k(for each k, respectively) that are parallel or have a common line. We can see that the degeneracy of the tetrahedron may be interpretated by $V \neq 0$ beautifully, by the division by zero.

10 Applications to Wazan geometry

For the sake of the great contributions to Wazan geometry by H. Okumura, we found new interesting results as applications of the division by zero calculus. We will introduce typical results, however, the results and their impacts will create some new fields in mathematics.

10.1 Circle and line

We will consider the fixed circle $x^2+(y-b)^2=b^2, b>0$. For a taching circle with this circle and the x axis is represented by

$$(x-2\sqrt{ab})^2+(y-a)^2=a^2.$$

Then, we have

$$\frac{x^2+y^2}{\sqrt{a}}-4\sqrt{b}x=2\sqrt{a}(y-2b)$$

and

$$\frac{x^2+y^2}{a}-4\sqrt{\frac{b}{a}}x=2(y-2b).$$

Then, by the division by zero, we have the reasonable results the origin, that is the point circle of the origin, the y axis and the line $y=2b$. (H. Okumura: 2017.10.13).

10.2 Three touching cirlces exteriously

For real numbers z, and $a,b>0$, the point $(0,2\sqrt{ab}/z)$ is denoted by V_z. H. Okumura and M. Watanabe gave the theorem in [34]:

Theorem 7. *The circle touching the circle* α: $(x-a)^2+y^2=a^2$ *and the circle* β: $(x+b)^2+y^2=b^2$ *at points different from the origin* O *and passing through* $V_{z\pm1}$ *is represented by*

$$\left(x-\frac{b-a}{z^2-1}\right)^2+\left(y-\frac{2z\sqrt{ab}}{z^2-1}\right)^2=\left(\frac{a+b}{z^2-1}\right)^2 \tag{10.1}$$

for a real number $z\neq\pm1$

The common external tangents of α *and* β *can be expressed by the equations*

$$(a-b)x\mp2\sqrt{ab}y+2ab=0. \tag{10.2}$$

Anyhow the authors give the exact representation with a parameter of the general circles touching with two circles touching each other. The common external tangent may be looked a circle touching for the general circles (as we know we can consider circles and lines as same ones in complex analysis or with the stereographic projection), however, they stated in the proof of the theorem that the common external tangents are obtained by the limiting $z \to \pm 1$. However, its logic will have a delicate problem.

Following our concept of the division by zero calculus, we will consider the case $z^2 = 1$ for the singular points in the general parametric representation of the touching circles.

10.2.1 Results

First, for $z = 1$ and $z = -1$, respectively by the division by zero calculus, we have from (10.1), surprisingly

$$x^2 + \frac{b-a}{2}x + y^2 \mp \sqrt{ab}y - ab = 0, \tag{10.3}$$

respectively.

Secondly, multiplying (10.1) by $(z^2 - 1)$, we immediately obtain surprisingly (10.2) for $z = 1$ and $z = -1$, respectively by the division by zero calculus.

In the usual way, when we consider the limiting $z \to \infty$ for (10.1), we obtain the trivial result of the point circle of the origin. However, the result may be obtained by the division by zero calculus at $w = 0$ by setting $w = 1/z$.

10.2.2 On the circle appeared

The circle (10.3) meets the circles α in two points

$$P_a\left(2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{a}{b}}\right), \quad Q_a\left(\frac{2ab}{9a+b}, -\frac{6a\sqrt{ab}}{9a+b}\right),$$

where $r_{\mathrm{A}} = ab/(a+b)$. Also it meet β in points

$$P_b\left(-2r_{\mathrm{A}}, 2r_{\mathrm{A}}\sqrt{\frac{b}{a}}\right), \quad Q_b\left(\frac{-2ab}{a+9b}, -\frac{6b\sqrt{ab}}{a+9b}\right).$$

The line P_aP_b is the common tangentials of the two circles α and β on the upper half plane. The lines P_aQ_a and P_bQ_b intersect at the point $R : \left(0, -\sqrt{ab}\right)$, which lies on the remaining tangencials of α and β. Furthermore, the circle (10.3) is orthogonal to the circle with center R passing through the origin.

10.3 The Descartes circle theorem

We recall the famous and beautiful theorem ([19, 53]):

Theorem (Descartes) *Let C_i $(i = 1, 2, 3)$ be circles touching to each other of radii r_i. If a circle C_4 touches the three circles, then its radius r_4 is given by*

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}}. \tag{10.4}$$

As well-known, circles and lines may be looked as the same ones in complex analysis, in the sense of stereographic projection and many reasons. Therefore, we will consider whether the theorem is valid for line cases and point cases for circles. Here, we will discuss this problem clearly from the division by zero viewpoint. The Descartes circle theorem is valid except for one case for lines and points for the three circles and for one exception case, we can obtain very interesting results, by the division by zero calculus.

We would like to consider all the cases for the Descartes theorem for lines and point circles, step by step.

10.3.1 One line and two circles case

We consider the case in which the circle C_3 is one of the external common tangents of the circles C_1 and C_2. This is a typical case in this paper. We assume $r_1 \geq r_2$. We now have $r_3 = 0$ in (10.4). Hence

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{0} \pm 2\sqrt{\frac{1}{r_1r_2} + \frac{1}{r_2 \cdot 0} + \frac{1}{0 \cdot r_1}} = \frac{1}{r_1} + \frac{1}{r_2} \pm 2\sqrt{\frac{1}{r_1r_2}}.$$

This implies

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

in the plus sign case. The circle C_4 is the incircle of the curvilinear triangle made by C_1, C_2 and C_3 (see Figure 1). In the minus sign case we have

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_2}} - \frac{1}{\sqrt{r_1}}.$$

In this case C_2 is the incircle of the curvilinear triangle made by the other three (see Figure 2).

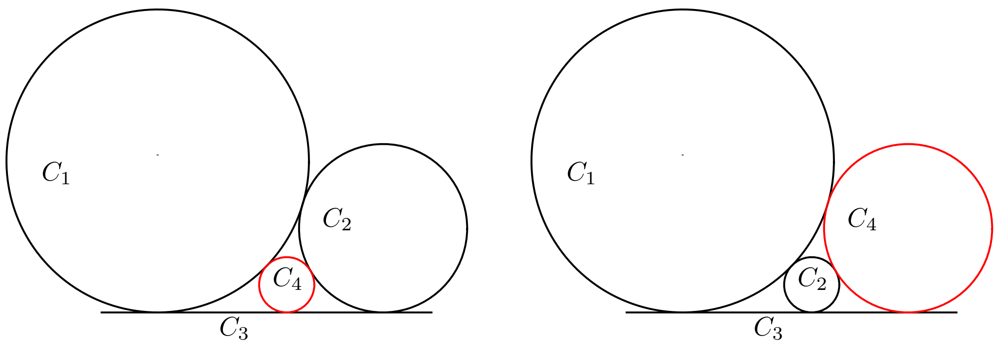

Figure 1. Figure 2.

Of course, the result is known. The result was also well-known in Wasan geometry [58] with the Decartes circle theorem itself.

10.3.2 Two lines and one circle case

In this case, the two lines have to be parallel, and so, this case is trivial, because then other two circles are the same size circles, by the division by zero $1/0 = 0$.

10.3.3 One point circle and two circles case

This case is another typical case for the theorem. Intuitively, for $r_3 = 0$, the circle C_3 is the common point of the circles C_1 and C_2. Then, there does not exist any touching circle of the three circles $C_j; j = 1, 2, 3$.

For the point circle C_3, we will consider it by limiting of circles attaching to the circles C_1 and C_2 to the common point. Then, we will examine the circles C_4 and the Descartes theorem.

In Theorem 7, by setting $z = 1/w$, we will consider the case $w = 0$; that is, the case $z = \infty$ in the classical sense; that is, the circle C_3 is reduced to the origin.

We look for the circles C_4 attaching with three circles $C_j; j = 1, 2, 3$. We set

$$C_4 : (x - x_4)^2 + (y - y_4)^2 = r_4^2. \tag{10.5}$$

Then, from the touching property we obtain:

$$x_4 = \frac{r_1 r_2 (r_2 - r_1) w^2}{D},$$

$$y_4 = \frac{2r_1 r_2 \left(\sqrt{r_1 r_2} + (r_1 + r_2) w\right) w}{D}$$

and

$$r_4 = \frac{r_1 r_2 (r_1 + r_2) w^2}{D},$$

where

$$D = r_1 r_2 + 2\sqrt{r_1 r_2}(r_1 + r_2) w + (r_1^2 + r_1 r_2 + r_2^2) w^2.$$

By inserting these values to (10.5), we obtain

$$f_0 + f_1 w + f_2 w^2 = 0,$$

where

$$f_0 = r_1 r_2 (x^2 + y^2),$$

$$f_1 = 2\sqrt{r_1 r_2}\big((r_1 + r_2)(x^2 + y^2) - 2r_1 r_2 y\big)$$

and

$$f_2 = (r_1^2 + r_1 r_2 + r_2^2)(x^2 + y^2) + 2r_1 r_2 (r_2 - r_1) x - 4(r_1 + r_2) y + 4r_1^2 r_2^2.$$

By using the division by zero calculus for $w = 0$, we obtain, for the first, for $w = 0$, the second by setting $w = 0$ after dividing by w and for the third case, by setting $w = 0$ after dividing by w^2,

$$x^2 + y^2 = 0, \tag{10.6}$$

$$(r_1 + r_2)(x^2 + y^2) - 2r_1 r_2 y = 0 \tag{10.7}$$

and

$$(r_1^2 + r_1 r_2 + r_2^2)(x^2 + y^2) + 2r_1 r_2 (r_2 - r_1) x - 4r_1 r_2 (r_1 + r_2) y + 4r_1^2 r_2^2 = 0. \tag{10.8}$$

Note that (10.7) is the red circle in Figure 3 and its radius is

$$\frac{r_1 r_2}{r_1 + r_2} \tag{10.9}$$

and (10.8) is the green circle in Figure 3 whose radius is

$$\frac{r_1 r_2 (r_1 + r_2)}{r_1^2 + r_1 r_2 + r_2^2}.$$

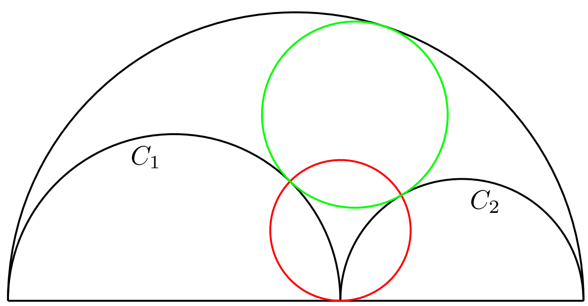

Figure 3.

When the circle C_3 is reduced to the origin, of course, the inscribed circle C_4 is reduced to the origin, then the Descartes theorem is not valid. However, by the division by zero calculus, then the origin of C_4 is changed suddenly for the cases (10.6), (10.7) and (10.8), and for the circle (10.7), the Descartes theorem is valid for $r_3 = 0$, surprisingly.

Indeed, in (9.4) we set $\xi = \sqrt{r_3}$, then (10.4) is as follows:

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\xi^2} \pm 2\frac{1}{\xi}\sqrt{\frac{\xi^2}{r_1 r_2} + \left(\frac{1}{r_1} + \frac{1}{r_2}\right)}.$$

and so, by the division by zero calculus at $\xi = 0$, we have

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2}$$

which is (10.9). Note, in particular, that the division by zero calculus may be applied in many ways and so, for the results obtained should be examined some meanings. This circle (10.7) may be looked a circle touching the origin and two circles C_1 and C_2, because by the division by zero calculus

$$\tan\frac{\pi}{2} = 0,$$

that is a popular property.

Meanwhile, the circle (10.8) is the attaching circle with the circles C_1, C_2 and the beautiful circle with center $((r_2-r_1),0)$ with radius r_1+r_2. The each of the areas surrounded by the three cicles C_1, C_2 and the circle of radius r_1+r_2 is called an arbelos, and the circle (10.7) is the famous Bankoff circle of the arbelos.

For $r_3=-(r_1+r_2)$, from the Descartes identity (10.4), we have (10.4). That is, when we consider that the circle C_3 is changed to the circle with center $((r_2-r_1),0)$ with radius r_1+r_2, the Descartes identity holds. Here, the minus sign shows that the circles C_1 and C_2 touch C_3 internally from the inside of C_3.

10.3.4 Two point circles and one circle case

This case is trivial, because, the exterior touching circle is coincident with one circle.

10.3.5 Three points case and three lines case

In these cases we have $r_j=0, j=1,2,3$ and the formula (10.4) shows that $r_4=0$. This statement is trivial in the general sense.

As the solution of the simplest equation

$$ax=b, \tag{10.10}$$

we have $x=0$ for $a=0, b\neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (10.10) is impossible. The zero will represent some **impossibility**.

In the Descartes theorem, three lines and three points cases, we can understand that the attaching circle does not exist, or it is the point and so the Descartes theorem is valid.

10.4 Circles and a chord

We recall the following result of the old Japanese geometry [57, 53, 34] (see Figure 4):

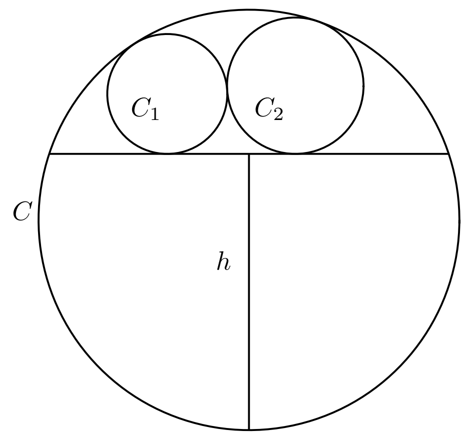

Figure 4.

Lemma 10. *Assume that the circle C with radius r is divided by a chord t into two arcs and let h be the distance from the midpoint of one of the arcs to t. If two externally touching circles C_1 and C_2 with radii r_1 and r_2 also touch the chord t and the other arc of the circle C internally, then h, r, r_1 and r_2 are related by*

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{r_1 r_2 h}}.$$

We are interesting in the limit case $r_1 = 0$ or $r_2 = 0$. In order to see the backgound of the lemma, we will see its simple proof.

The centers of C_1 and C_2 can be on the opposite sides of the normal dropped on t from the center of C or on the same side of this normal. From the right triangles formed by the centers of C and C_i $(i = 1, 2)$, the line parallel to t through the center of C, and the normal dropped on t from the center of C_i, we have

$$|\sqrt{(r - r_1)^2 - (h + r_1 - r)^2} \pm \sqrt{(r - r_2)^2 - (h + r_2 - r)^2}| = 2\sqrt{r_1 r_2},$$

where we used the fact that the segment length of the common external tangent of C_1 and C_2 between the tangency points is equal to $2\sqrt{r_1 r_2}$. The formula of the lemma follows from this equation.

10.4.1 Results

We introduce the coordinates in the following way: the bottom of the circle C is the origin and tangential line at the origin of the circle C is the x axis and the y axis is given as in the center of the circle C is $(0, r)$. We denote the centers of the circles $C_j; j = 1, 2$ by (x_j, y_j), then we have

$$y_1 = h + r_1, \quad y_2 = h + r_2.$$

Then, from the attaching conditions, we obtain the three equations:

$$(x_2 - x_1)^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2,$$
$$x_1^2 + (h - r + r_1)^2 = (r - r_1)^2$$

and

$$x_2^2 + (h - r + r_2)^2 = (r - r_2)^2.$$

Solving the equations for x_1, x_2 and r_2, we get four sets of the solutions. Let $h = 2r_3$, $v = r - r_1 - r_3$. Then two sets are:

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \pm 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

The other two sets are

$$\begin{aligned} x_1 &= \pm 2\sqrt{r_3 v}, \\ x_2 &= \mp 2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

We now consider the solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3 v}, \\ x_2 &= 2\frac{r_1\sqrt{rr_3} + r_3\sqrt{r_3 v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1 r_3(2\sqrt{r}(\sqrt{r} - \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{g_0 + g_1 r_1 + g_2 r_1^2 + g_3}{(r_1 + r_3)^2},$$

where

$$g_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$

$$g_1 = 2r_3((x - \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$

$$g_2 = (x - 2\sqrt{rr_3})^2 + y^2 - 2r_3y,$$

and

$$g_3 = 4r_3\sqrt{v}(r_1(\sqrt{r}y - \sqrt{r_3}x) - r_3\sqrt{r_3}x).$$

We now consider another solution

$$\begin{aligned} x_1 &= 2\sqrt{r_3v}, \\ x_2 &= -2\frac{r_1\sqrt{rr_3} - r_3\sqrt{r_3v}}{r_1 + r_3}, \\ r_2 &= \frac{r_1r_3(2\sqrt{r}(\sqrt{r} + \sqrt{v}) - (r_1 + r_3))}{(r_1 + r_3)^2}. \end{aligned}$$

Then

$$(x - x_2)^2 + (y - y_2)^2 - r_2^2 = \frac{k_0 + k_1r_1 + k_2r_1^2 + k_3}{(r_1 + r_3)^2},$$

where

$$k_0 = r_3^2(x^2 + y(y - 4r_3) + 4rr_3),$$

$$k_1 = 2r_3((x + \sqrt{rr_3})^2 + y^2 - (2r + 3r_3)y + 3rr_3),$$

$$k_2 = (x + 2\sqrt{rr_3})^2 + y^2 - 2r_3y,$$

and

$$k_3 = -4r_3\sqrt{v}(r_1(\sqrt{r}y + \sqrt{r_3}x) + r_3\sqrt{r_3}x).$$

We thus see that the circle C_2 is represented by

$$(g_0 + g_3) + g_1r_1 + g_2r_1^2 = 0$$

and

$$(k_0 + k_3) + k_1r_1 + k_2r_1^2 = 0.$$

For the symmetry, we consider only the above case. We obtain the division by zero calculus, first by setting $r_1 = 0$, the next by setting $r_1 = 0$ after dividing by r_1 and the last by setting $r_1 = 0$ after dividing by r_1^2,

$$g_0 + g_3 = 0,$$

$$g_1 = 0,$$

and

$$g_2 = 0.$$

That is,

$$\left(x - \sqrt{2rh - h^2}\right)^2 + (y-h)^2 = 0,$$

$$\left(x - \sqrt{\frac{rh}{2}}\right)^2 + \left(y - \left(r + \frac{3h}{4}\right)\right)^2 = r^2 + \frac{9}{16}h^2,$$

and

$$\left(x - \sqrt{2rh}\right)^2 + \left(y - \frac{h}{2}\right)^2 = \left(\frac{h}{2}\right)^2.$$

The first equation represents one $(\sqrt{2rh - h^2}, h)$ of the points of intersection of the circle C and the chord t (see Figure 5). The second equation expresses the red circle in the figure. The third equation expresses the circle touching C externally, the x-axis and the extended chord t denoted by the green circle in the figure. The last two circles are orthogonal to the circle with center origin passing through the points of intersection of C and t.

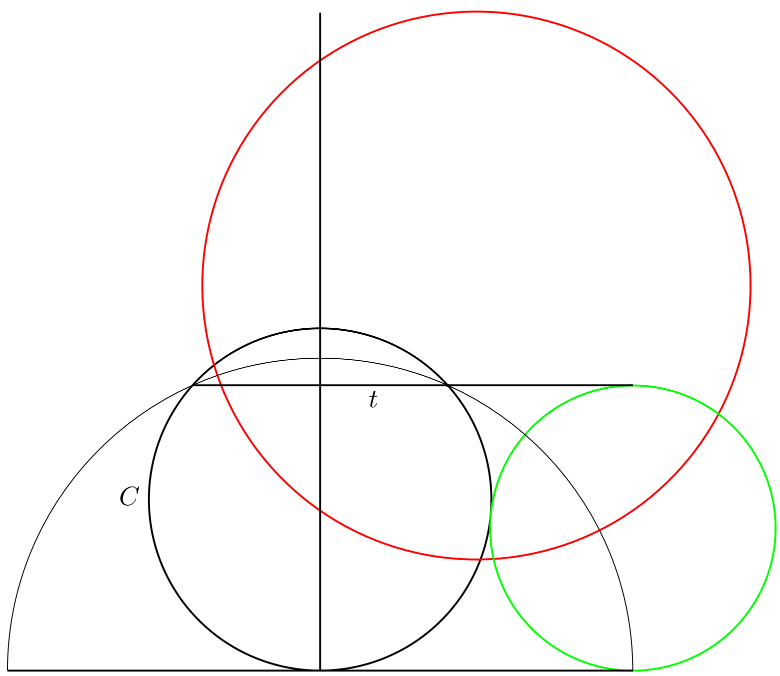

Figure 5

Now for the beautiful identity in the lemma, for $r_1 = 0$, we have, by the division by zero,

$$\frac{1}{0} + \frac{1}{r_2} + \frac{2}{h} = 2\sqrt{\frac{2r}{0 \cdot r_2 h}}$$

and

$$r_2 = -\frac{h}{2}.$$

Here, the minus sigh will mean that the blue circle is attaching with the circle C in the outside of the circle C; that is, we can consider that when the circle C_1 is reduced to the point $(\sqrt{2rh - h^2}, h)$, then the circle C_2 is suddenly changed to the blue circle and the beautiful identity is still valid. Note, in particular, the brue circle is attaching with the circle C and the cord t.

Meanwhile, for the curious red circle, we do not know its property, however, we know curiously that it is orthogonal with the circle with the center at the origin and with radius $\sqrt{2rh}$ passing through the points $(\pm\sqrt{2rh - h^2}, h)$.

This subsection is based on the paper [38].

11 Introduction of formulas $\log 0 = \log \infty = 0$

For any fixed complex number a, we will consider the sector domain $\Delta_a(\alpha, \beta)$ defined by

$$0 \leq \alpha < \arg(z - a) < \beta < 2\pi$$

on the complex z plane and we consider the conformal mapping of $\Delta_a(\alpha, \beta)$ into the complex W plane by the mapping

$$W = \log(z - a). \tag{11.1}$$

Then, the image domain is represented by

$$S(\alpha, \beta) = \{W; \alpha < \Im W < \beta\}.$$

Two lines $\{W; \Im W = \alpha\}$ and $\{W; \Im W = \beta\}$ usually were considered as having the common point at infinity, however, in the division by zero, the point is represented by zero.

Therefore, $\log 0$ and $\log \infty$ **should be defined as zero**. Here, $\log \infty$ is precisely given in the sense of $[\log z]_{z=\infty}$. However, the properties of the logarithmic function should not be expected more, we should consider the value only. For example,

$$\log 0 = \log(2 \cdot 0) = \log 2 + \log 0$$

is not valid.

In particular, in many formulas in physics, in some expression, for some constants A, B

$$\log \frac{A}{B},$$

if we consider the case that A or B is zero, then we should consider it in the form

$$\log \frac{A}{B} = \log A - \log B, \tag{11.2}$$

and we should put zero in A or B. Then, in many formulas, we will be able to consider the case that A or B is zero. For the case that A or B is zero, the identity (11.1) is not valid, then the expression $\log A - \log B$ may be valid in many physical formulas. However, the results are case by case, and we should check the obtained results for applying the formula (11.1) for $A = 0$ or $B = 0$.

11.1 Applications of $\log 0 = 0$

We can apply the result $\log 0 = 0$ for many cases as in the following way.

For example, we will consider the differential equation

$$xy' = xy^2 - a^2 x \log^{2k}(\beta x) + ak \log^{k-1}(\beta x). \tag{11.3}$$

For the solution $y = a\log^{2k}(\beta x)$([41], page 95, **5**), we can consider the solution $y = 0$ as $\beta = 0$.

In the famous function (Leminiscate)

$$x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}, \quad a > 0, \tag{11.4}$$

we have

$$x = a \log \left[\frac{a + \sqrt{a^2 - y^2}}{y} \exp\left(-\frac{1}{a}\sqrt{a^2 - y^2} \right) \right]. \tag{11.5}$$

By the division by zero, at the point $y = 0$

$$\left[\frac{a + \sqrt{a^2 - y^2}}{y} \exp\left(-\frac{1}{a}\sqrt{a^2 - y^2} \right) \right] = 0. \tag{11.6}$$

Thus the curve passes also the origin (0.0).

In the differential equation

$$x^2 y''' + 4x^2 y'' - 2xy' - 4y = \log x, \tag{11.7}$$

we have the general solution

$$y = \frac{C_1}{x} + \frac{C_2}{x^2} + C_3 x^2 - \frac{1}{4}\log x + \frac{1}{4}, \tag{11.8}$$

satisfying that at the origin $x = 0$

$$y(0) = \frac{1}{4}, y'(0) = 0, y''(0) = 2C_3, y'''(0) = 0. \tag{11.9}$$

We can give the values C_1 and C_2. For the sake of the division by zero, we can, in general, consider differential equations even at analytic and isolated singular points.

In the formula ([14], page 153), for $0 \leq x, t \leq \pi$

$$\sum_{n=1}^{\infty} \frac{\sin ns \sin nt}{n} = \frac{1}{2} \log |\sin((s+t)/2)/\sin((s-t)/2)|, \tag{11.10}$$

for $s = t = 0, \pi$, we can interprete that

$$0 = \frac{1}{2} \log \frac{0}{0} = \log 0. \tag{11.11}$$

In general, for $s = t$, we may consider that

$$\sum_{n=1}^{\infty} \frac{\sin^2 ns}{n} = \frac{1}{2} \log |\sin((s+s)/2)/0)| \tag{11.12}$$

$$= \frac{1}{2} \log |\sin ns/0| = \frac{1}{2} \log 0 = 0.$$

Note that this result is not a contradiction. Recall the case of the function $y = 1/x$ at the origin:

$$\lim_{x \to +0} \frac{1}{x} = +\infty, \tag{11.13}$$

in the monotonically increasing way, however,

$$\left[\frac{1}{x}\right]_{x=0} = 0. \tag{11.14}$$

Such a discontinuity property is important in the division by zero.

We will give a physical sense of $\log 0 = 0$. We shall consider a uniform line density μ on the $z-$ axis, then the force field $\mathbf{F}$ and the potential ϕ are given, for $\mathbf{p} = x\mathbf{i} + y\mathbf{j}, p = |\mathbf{p}|$,

$$\mathbf{F} = -\frac{2\mu}{p^2}\mathbf{p} \tag{11.15}$$

and

$$\phi = -2\mu \log \frac{1}{p}, \tag{11.16}$$

respectively. On the z- axis, we have, of course,

$$\mathbf{F} = \mathbf{0}, \phi = 0. \tag{11.17}$$

11.2 Robin constant and Green's functions

From the typical case, we will consider a fundamental application. Let $D(a, R) = \{|z| > R\}$ be the outer disc on the complex plane. Then, the Riemann mapping function that maps conformally onto the unit disc $\{|W| < 1\}$ and the point at infinity to the origin is given by

$$W = \frac{R}{z-a}. \tag{11.18}$$

Therefore, the Green function $G(z, \infty)$ of $D(a, R)$ is given by

$$G(z, \infty) = -\log\left\{\frac{R}{|z-a|}\right\}. \tag{11.19}$$

Therefore, from the representation

$$G(z, \infty) = -\log R + \log|z| + \log\left(1 - \frac{a}{|z|}\right), \tag{11.20}$$

we have the identity

$$G(\infty, \infty) = -\log R, \tag{11.21}$$

that is the Robin constant of $D(a, R)$. This formula is valid in the general situation, because the Robin constant is defined by

$$\lim_{z \to b}\{G(z, b) + \log|z-b|\}, \tag{11.22}$$

for a general Green function with pole at b of some domain ([2]).

11.3 $e^0 = 1, 0$

By the introduction of the value $\log 0 = 0$, as the inversion function $y = e^x$ of the logarithmic function, we will consider that $y = e^0 = 0$. Indeed, we will show that this definition is very natural.

We will consider the conformal mapping $W = e^z$ of the strip

$$S(-\pi i, \pi i) = \{z; -\pi < \Im z < \pi\}$$

onto the whole W plane cut by the negative real line $(-\infty, 0]$. Of course, the origin 0 corresponds to 1. Meanwhile, we see that the negative line $(-\infty, 0]$

corresponds to the negative real line $(-\infty, 0]$. In particular, on the real line $\lim_{x\to-\infty} e^x = 0$. In our new space idea from the division by zero, the point at infinity is represented by zero and therefore, we should define as

$$e^0 = 0. \tag{11.23}$$

For the fundamental exponential function $W = \exp z$, at the origin, we should consider 2 valued function. The value 1 is the natural value as a regular point of analytic function, meanwhile the value 0 is given with a strong discontinuity; however, this value will appear in the universe as a natural way.

For the elementary functions $y = x^n, n = \pm 1, \pm 2, \cdots$, we have

$$y = e^{n \log x}. \tag{11.24}$$

Then, we wish to have

$$y(0) = e^{n \log 0} = e^0 = 0. \tag{11.25}$$

As a typical example, we will consider the simple differential equation

$$\frac{dx}{x} - \frac{2ydy}{1+y^2} = 0. \tag{11.26}$$

Then, by the usual method,

$$\log |x| - \log |1 + y^2| = C; \tag{11.27}$$

that is,

$$\log \left| \frac{x}{1+y^2} \right| = \log e^C = \log K, K = e^C > 0 \tag{11.28}$$

and

$$\frac{x}{1+y^2} = \pm K. \tag{11.29}$$

However, the constant K may be taken zero, as we see directly $\log e^C = \log K = 0$.

In the differential equations

$$y' = -\lambda e^{\lambda x} y^2 + a e^{\mu x} y - a e^{(\mu-\lambda)x} \tag{11.30}$$

and

$$y' = -be^{\mu x}y^2 + a\lambda e^{\lambda x}y - a^2 be^{(\mu+2\lambda)x} \tag{11.31}$$

we have solutions

$$y = -e^{-\lambda x}, \tag{11.32}$$

$$y = ae^{\lambda x}, \tag{11.33}$$

respectively. For $\lambda = 0$, as $y = -1$, $y = a$ are solutions, respectively, however, the functions $y = 0$, $y = 0$ are not solutions, respectively. However, many and many cases, as the function $y = e^{0 \cdot x} = 0$, we see that the function is solutions of differential equations, when $y = e^{\lambda \cdot x}$ is the solutions. See [41] for many concrete examples.

Meanwhile, we will consider the Fourier integral

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-\alpha|t|} dt = \frac{2\alpha}{\alpha^2 + \omega^2}. \tag{11.34}$$

For the case $\alpha = 0$, if this formula valid, then we have to consider $e^0 = 0$.

Furthermore, by Poisson's formula, we have

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + (2\pi n)^2}. \tag{11.35}$$

If $e^0 = 0$, then the above identity is still valid, however, for $e^0 = 1$, the identity is not valid. We have many examples.

For the integral

$$\int_0^{\infty} \frac{x^3 \sin(ax)}{x^4 + 4} dx = \frac{\pi}{2} e^{-a} \cos a, \tag{11.36}$$

the formula is valid for $a = 0$.

For the integral

$$\int_0^{\infty} \frac{\xi \sin(x\xi)}{1 + a^2 \xi} d\xi = \frac{\pi}{2a^2} e^{-(x/a)}, \quad x > 0, \tag{11.37}$$

the formula is valid for $x = 0$.

11.4 $0^0 = 1, 0$

By the standard definition, we will consider

$$0^0 = \exp(0 \log 0) = \exp 0 = 1, 0. \tag{11.38}$$

The value 1 is famous which was derived by N. Abel, meanwhile, H. Michiwaki have directly derived it as 0 from the result of the division by zero. However, we now know that $0^0 = 1, 0$ is the natural result.

We will see its reality.

For $0^0 = 1$**:**

In general, for $z \neq 0$, from $z^0 = e^{0 \log z}$, $z^0 = 1$, and so, we will consider that $0^0 = 1$ in a natural way.

For example, in the elementary expansion

$$(1+z)^n = \sum_{k=0}^{n} {}_nC_k z^k \tag{11.39}$$

the formula $0^0 = 1$ will be convenient for $k = 0$ and $z = 0$.

In the fundamental definition

$$\exp z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \tag{11.40}$$

in order to have a sense of the expansion at $z = 0$ and $k = 0$, we have to accept the formula $0^0 = 1$.

In the differential formula

$$\frac{d^n}{dx^n} x^n = n x^{n-1}, \tag{11.41}$$

in the case $n = 1$ and $x = 0$, the formula $0^0 = 1$ is convenient and natural.

In the Laurent expansion (5.5), if $0^0 = 1$, it may be written simply as

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n,$$

for $f(a) = C_0$.

For $0^0 = 0$**:**

For any positive integer n, since $z^n = 0$ for $z = 0$, we wish to consider that $0^0 = 0$ for $n = 0$.

11.5 $\cos 0 = 1, 0$

Since

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \tag{11.42}$$

we wish to consider also the value $\cos 0 = 0$.

The values $e^0 = 0$ and $\cos 0 = 0$ may be considered that the values at the point at infinity are reflected to the origin and other many functions will have the same property.

The short version of this section was given by [24] in the Proceedings of the International Conference: https://sites.google.com/site/sandrapinelas/icddea-2017

11.6 Finite parts of Hadamard in singular integrals

Singular integral equations are presently encountered in a wide range of mathematical models, for instance in acoustics, fluid dynamics, elasticity and fracture mechanics. Together with these models, a variety of methods and applications for these integral equations has been developed. See, for example, [12, 16, 28, 30].

For the numerical calculation of this finite part, see [35], and there, they gave an effective numerical formulas by using the DE formula. See also its references for various methods.

For singular integrals, we will consider their integrals as divergence, however, the Haramard finite part or Cauhy's principal values give finite values; that is, from divergence values we will consider finite values; for this interesting property, we will be able to give an natural interpretation by the division by zero calculus.

Let $F(x)$ be an integrable function on an interval (c, d). The functions $F(x)/(x-a)^n (n = 1, 2, 3..., c < a < d)$ are, in general, not integrable on (c, d). However, for any $\epsilon > 0$, of course, the functions

$$\left(\int_c^{a-\epsilon} + \int_{a+\epsilon}^d\right) \frac{F(x)}{(x-a)^n} dx \tag{11.43}$$

are integrable. For an integrable function $\varphi(x)$ on (a, d), we assume the Taylor expansion

$$F(x) = \sum_{k=0}^{n-1} \frac{F^{(k)}(a)}{k!}(x-a)^k + \varphi(x)(x-a)^n. \tag{11.44}$$

Then, we have

$$\int_{a+\epsilon}^{d} \frac{F(x)}{(x-a)^n}dx$$

$$= \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{\epsilon^{n-k-1}} - \frac{F^{(n-1)}(a)}{(n-1)!} \log \epsilon$$

$$+ \left\{ - \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{(d-a)^{n-k-1}} + \frac{F^{(n-1)}(a)}{(n-1)!} \log(d-a) + \int_{a+\epsilon}^{d} \varphi(x)dx \right\}.$$

Then, the last term {....} is the finite part of Hadamard of the integral

$$\int_{a}^{d} \frac{F(x)}{(x-a)^n}dx \tag{11.45}$$

and is written by

$$\text{f. p.} \int_{a}^{d} \frac{F(x)}{(x-a)^n}dx; \tag{11.46}$$

that is, precisely

$$\text{f. p.} \int_{a}^{d} \frac{F(x)}{(x-a)^n}dx$$

$$:= \lim_{\epsilon \to +0} \left\{ \int_{a+\epsilon}^{d} \frac{F(x)}{(x-a)^n}dx - \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1}{\epsilon^{n-k-1}} + \frac{F^{(n-1)}(a)}{(n-1)!} \log \epsilon \right\}. \tag{11.47}$$

We do not take the limiting $\epsilon \to +0$, but we put $\epsilon = 0$, in (11.47), then we obtain, by the division by zero calculus:

$$\text{f. p.} \int_{a}^{d} \frac{F(x)}{(x-a)^n}dx = \int_{a}^{d} \frac{F(x)}{(x-a)^n}dx. \tag{11.48}$$

Division by zero will give the natural meaning (**definition**) for the above two integrals.

Of course,

$$\text{f. p.} \int_{c}^{d} \frac{F(x)}{(x-a)^n}dx := \text{f. p.} \int_{c}^{a} \frac{F(x)}{(x-a)^n}dx + \text{f. p.} \int_{a}^{d} \frac{F(x)}{(x-a)^n}dx. \tag{11.49}$$

When $n = 1$, the integral is the Cauchy principal value.

In particular, for the expression (11.47), we have, missing $\log \epsilon$ term, for $n \geq 2$

$$\text{f. p.} \int_c^d \frac{F(x)}{(x-a)^n} dx$$

$$= \lim_{\epsilon \to +0} \left\{ \left(\int_c^{a-\epsilon} + \int_{a+\epsilon}^d \right) \frac{F(x)}{(x-a)^n} dx - \sum_{k=0}^{n-2} \frac{F^{(k)}(a)}{k!(n-k-1)} \frac{1+(-1)^{n-k}}{\epsilon^{n-k-1}} \right\}. \tag{11.50}$$

The content of this section was given in the paper [24].

12 Basic meanings of values at isolated singular points of analytic functions

Since the values of analytic functions with isolated singular points were given by the coefficients C_0 of the Laurent expansions (the first coefficients of the regular parts) as the division by zero calculus. Therefore, their characteristic property may be considered as arbitrary ones by any sift of the image complex plane. Therefore, we can consider the values as zero in any Laurent expansions by shifts, as normalizations. However, if by another normalizations, the Laurent expansions are determind, then the values will have their senses. We will examine such properties for the Riemann mapping function.

Let D be a simply-connected domain containing the point at infinity having at least two boundary points. Then, by the celebrated theorem of Riemann, there exists a uniquely determined conformal mapping with a series expansion

$$W = f(z) = C_1 z + C_0 + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots, C_1 > 0, \tag{12.1}$$

at the point at infinity which maps the domain D onto the exterior $|w| > 1$ of the unit disc in the complex W plane. We can normalize (12.1) as follows:

$$\frac{f(z)}{C_1} = z + \frac{C_0}{C_1} + \frac{C_{-1}}{C_1 z} + \frac{C_{-2}}{C_1 z^2} + \ldots. \tag{12.2}$$

Then, this function $\frac{f(z)}{C_1}$ maps D onto the exterior of a circle of radius $1/C_1$ and so, it is called the **mapping radius** of D. See [5, 56]. Meanwhile, from the normalization

$$f(z) - C_0 = C_1 z + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \ldots, \tag{12.3}$$

by the natural shift C_0 of the image plane, the unit circle is mapped to the unit circle with center C_0. Therefore, C_0 may be called as **mapping center** of D. The function $f(z)$ takes the value C_0 at the point at infinity in the sense of the division by zero and now we have its natural sense by the mapping center of D. We have considered the value of the function $f(z)$ as infinity at the point at infinity, however, practically it was the value C_0. This will mean that in a sense the value C_0 is the most far point from the point at infinity or the image domain with the strong discontinuity.

The properties of mapping radius were investigated deeply in conformal mapping theory like estimations, extremal properties and meanings of the values, however, it seems that there is no information on the property of mapping center. See many books on conformal mapping theory or analytic function theory. See [56] for example.

12.1 Values of typical Laurent expansions

The values at singular points of analytic functions are represented by the integrals, and so for given functions, the calculations will be simple numerically, however, their analytical (precise) values will be given by using the known Taylor or Laurent expansions. In order to obtain some feelings for the values at singular points of analytic functions, we will see typical examples and fundamental properties.

For

$$f(z) = \frac{1}{\cos z - 1}, \quad f(0) = -\frac{1}{6}. \tag{12.4}$$

For

$$f(z) = \frac{\log(1+z)}{z}, \quad f(0) = \frac{-1}{2}. \tag{12.5}$$

For

$$f(z) = \frac{1}{z(z+1)}, \quad f(0) = -1. \tag{12.6}$$

Here, note that on $|z| > 1$,

$$f(z) = \frac{1}{z(z+1)} = \frac{1}{z^2} - \frac{1}{z^3} + \frac{1}{z^4} - ..., \quad f(0) = 0. \tag{12.7}$$

That is, in Theorem 1, when a is an isolated singular point, we have to consider the Laurent expansion on $\{0 < r < |z - a| < R\}$ such that r may be taken arbitrary small r, because we are considering the function at a.

For

$$f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}, \quad f(i) = \frac{1}{4}. \tag{12.8}$$

For

$$f(z) = \frac{1}{\sqrt{(z+1)} - 1}, \quad f(0) = \frac{1}{2}. \tag{12.9}$$

From the well-known expansion ([1], page 807) of the Riemann zeta function

$$\zeta(s) = \frac{1}{s-1} + \gamma - \gamma_1(s-1) + \gamma_2(s-1)^2 + ..., \tag{12.10}$$

we see that the Euler constant γ is the value at $s = 1$:

$$\zeta(1) = \gamma. \tag{12.11}$$

From the representation of the Gamma function $\Gamma(z)$

$$\Gamma(z) = \int_1^\infty e^{-t} t^{z-1} dt + \sum_{n=0}^\infty \frac{(-1)^n}{n!(z+n)} \tag{12.12}$$

([42], page 472), we have

$$\Gamma(-m) = E_{m+1}(1) + \sum_{n=0, n\neq m}^\infty \frac{(-1)^n}{n!(-m+n)}$$

and

$$[\Gamma(z)\cdot(z+n)](-n) = \frac{(-1)^n}{n!}.$$

In particular, we obtain

$$\Gamma(0) = -\gamma$$

([1], 229 p. (5.1.11)).

We can consider many special functions and the values at singular points. For example,

$$Y_{3/2}(z) = J_{-3/2}(z) = -\sqrt{\frac{2}{\pi z}}\left(\sin z + \frac{\cos z}{z}\right),$$

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sinh z,$$

$$K_{1/2}(z) = K_{-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z},$$

and so on. They take the value zero at the origin, however, we can consider some meanings of the value.

Of course, the product property is, in general, not valid:

$$f(0)\cdot g(0) \neq (f(z)g(z))(0); \tag{12.13}$$

indeed, for the functions $f(z) = z + 1/z$ and $g(z) = 1/z + 1/(z^2)$

$$f(0) = 0, g(0) = 0, (f(z)g(z))(0) = 1. \tag{12.14}$$

For an analytic function $f(z)$ with a zero point a, for the inversion function

$$(f(z))^{-1} := \frac{1}{f(z)},$$

we can calculate the value $(f(a))^{-1}$ at the singular point a.

For example, note that: for the function

$$f(z) = z - \frac{1}{z},$$

$f(0) = 0, f(1) = 0$ and $f(-1) = 0$. Then, we have

$$(f(z))^{-1} = \frac{1}{2(z+1)} + \frac{1}{2(z-1)}.$$

Hence,

$$((f(z))^{-1})(z = 0) = 0, ((f(z))^{-1})(z = 1) = \frac{1}{4}, ((f(z))^{-1}(z = -1) = -\frac{1}{4}.$$

Here, note that the point $z = 0$ is not a regular point of the function $f(z)$.

We, meanwhile, obtain that

$$\left(\frac{1}{\log x}\right)_{x=1} = 0. \tag{12.15}$$

Indeed, we consider the function $y = \exp(1/x), x \in \mathbf{R}$ and its inverse function $y = \frac{1}{\log x}$. By the symmetric of the two functions with respect to the function $y = x$, we have the desired result.

Here, note that for the function $\frac{1}{\log x}$, we cannot use the Laurent expansion around $x = 1$, and therefore, the result is not trivial.

We shall refer to the trigonometric functions. See, for example, ([13], page 75) for the expansions.

From the expansion

$$\frac{1}{\sin z} = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty} (-1)^{\nu} \left(\frac{1}{z - \nu\pi} + \frac{1}{\nu\pi}\right), \tag{12.16}$$

$$\left(\frac{1}{\sin z}\right)(0) = 0.$$

Meanwhile, from the expansion

$$\frac{1}{\sin^2 z} = \sum_{\nu=-\infty}^{\infty} \frac{1}{(z-\nu\pi)^2}, \tag{12.17}$$

$$\left(\frac{1}{\sin^2 z}\right)(0) = \frac{2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{1}{3}.$$

From the expansion

$$\frac{1}{\cos z} = 1 + \sum_{\nu=-\infty}^{+\infty}(-1)^{\nu}\left(\frac{1}{z-(2\nu-1)\pi/2} + \frac{2}{(2\nu-1)\pi}\right), \tag{12.18}$$

$$\left(\frac{1}{\cos z}\right)\left(\frac{\pi}{2}\right) = 1 - \frac{4}{\pi}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{2\nu+1} = 0.$$

Meanwhile, from the expansion

$$\frac{1}{\cos^2 z} = \sum_{\nu=-\infty}^{+\infty}\frac{1}{(z-(2\nu-1)\pi/2)^2}, \tag{12.19}$$

$$\left(\frac{1}{\cos^2 z}\right)\left(\frac{\pi}{2}\right) = \frac{2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{1}{3}.$$

12.2 Values of domain functions

In this section, we will examine hidden values of typical domain functions. For a basic reference, see [31].

1). For the mapping

$$W = \frac{z}{1-z} \tag{12.20}$$

that maps conformally the unit disc $|z| < 1$ onto the half-plane $\{ReW > \frac{1}{2}\}$, we have

$$W(1) = -1. \tag{12.21}$$

2). For the Koebe function

$$W = \frac{z}{(1-z)^2} \tag{12.22}$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of $(-\infty, -\frac{1}{4})$ we have

$$W(1) = 0. \tag{12.23}$$

We can understand it as follows: the boundary point $z = 1$ of the unit disc is mapped to the infinity point, however, the point is connected to the origin. We can see the similar property, for many cases.

3). For the Joukowsky transform

$$W = \frac{1}{2}\left(\frac{1}{z} + z\right) \tag{12.24}$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of $[-1, 1]$ we have

$$W(0) = 0. \tag{12.25}$$

This correspondence will be curious in a sense. The interior point the origin corresponds to the boundary point of the origin. Should we consider the inverse of the case 2)? - the image may be connected to the origin.

4). For the transform

$$W = \frac{z}{1-z^2} \tag{12.26}$$

that maps conformally the unit disc $|z| < 1$ onto the cut plane of the imaginary axis of $[+\infty, i/2]$ and $[-\infty, -i/2]$ we have

$$W(1) = -\frac{1}{4}, \quad W(-1) = \frac{1}{4}, \tag{12.27}$$

by the method of Laurent expansion method, curiously. Should we consider the values at $z = 1$ and $z = -1$ as 0 from $1/0$ and $-1/0$ by the insertings $z = 1$ and $z = -1$ in the numerator and denominator?

5). For the conformal mapping $W = P(z; 0, v), |v| < 1$ of the unit disc onto the circular slit W plane that is normalized by $P(0; 0, v) = 0$ and

$$P(z : 0, v) = \frac{1}{z - v} + C_0 + C_0(z - v) + ..., \tag{12.28}$$

is given by, explicitly

$$P(z;0,v) = \frac{1}{v(1-|v|^2)} \frac{z(1-\overline{v}z)}{z-v} \tag{12.29}$$

([31], 340 page). Then, we obtain

$$P(z:0,v)|_{z=v} = C_0 = \frac{1-2|v|^2}{v(1-|v|^2)}, \tag{12.30}$$

at $z = v$ by the Laurent expansion method. By the constant C_0, we can interpret as in the mapping center by shift of the image plane. We may also give the value for $z = v$ by

$$P(z:0,v)|_{z=v} = \left[\frac{1}{v(1-|v|^2)} \frac{z(1-\overline{v}z)}{z-v}\right]_{z=v} = \frac{v(1-|v|^2)}{0} = 0. \tag{12.31}$$

The circumstance is similar for the corresponding canonical conformal mapping $Q(z:0,v)$ for the radial slit mapping.

12.3 Mysterious properties at the point at infinity

In this subsection, we will refer to some feelings on the point at infinity, because, the division by zero creates a new world on the point at infinity.

12.3.1 Many points at infinity?

When we consider a circle with center P, by the inversion with respect to the circle, the points of a neighborhood at the point P are mapped to a neighborhood around the point at infinity except the point P. This property is independent of the radius of the circle. It looks that the point at infinity is depending on the center P. This will mean that there exist many points at infinity, in a sense.

12.3.2 Stereographic projection

The point at infinity may be realized by the stereographic projection as well known. However, the projection is dependent on the position of the sphere (the plane coordinates). Does this mean that there exist many points at infinity?

12.3.3 Laurent expansion

From the definition of the division by zero calculus, we see that if there exists a negative n term in (5.5)

$$\lim_{z\to a} f(z) = \infty,$$

however, we have (5.6). The values at the point a have many values, that are all complex numbers. At least, **in this sense**, we see that we have many points at the point of infinity.

In the sequel, we will show typical points at infinity.

12.3.4 Diocles' curve of Carystus (BC 240? - BC 180?)

The beautiful curve

$$y^2 = \frac{x^3}{2a - x}, \quad a > 0$$

is considered by Diocles. By setting $X = \sqrt{2a - x}$ we have

$$y = \pm\frac{x^{(3/2)}}{\sqrt{2a - x}} = \pm\frac{(2a - X^2)^{(3/2)}}{X}.$$

Then, by the division by zero calculus at $X = 0$, we have a reasonable value 0.

Meanwhile, for the function $\frac{x^3}{2a-x}$, we have $-12a^2$, by the division by zero calculus at $x = 2a$. This leads to a wrong value.

12.3.5 Nicomedes' curve (BC 280 - BC 210)

The very interesting curve

$$r = a + \frac{b}{\cos\theta}$$

is considered by Nicomedes from the viewpoint of the 1/3 division of an angle. That has very interesting geometrical meanings. For the case $\theta = \pm(\pi/2)$, we have $r = a$, by the division by zero calculus.

Of course, the function is symmetric for $\theta = 0$, however, we have a mysterious value $r = a$, for $\theta = \pm(\pi/2)$. Look the beautiful graph of the function.

12.3.6 Newton's curve (1642 - 1727)

Meanwhile, for the famous Newton curve

$$y = ax^2 + bx + c + \frac{d}{x} \quad (a, d \neq 0),$$

of course, we have $y(0) = c$.

Meanwhile, in the division by zero calculus, the value is determined by the information around any analytical point for an analytic function, as we see from the basic property of analytic functions.

At this moment, the properties of the values of analytic functions at isolated singular points are mysterious, in particular, in the geometrical sense.

12.3.7 Unbounded, however, bounded

We will consider the high

$$y = \tan\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

on the line $x = 1$. Then, the high y is unbounded, however, the high line (gradient) can not be extended beyond the y axis. The restriction is given by $0 = \tan(\pi/2)$.

Recall the stereographic projection of the complex plane. The points on the plane can be expanded in an unbounded way, however, all the points on the complex plane have to be corresponded to the points of the Riemann sphere. The restriction is the point at infinity which corresponds to the north pole of the Riemann sphere and the point at infinity is represented by 0.

This subsection is represented in [51].

13 Division by zero calculus on mutidimensional spaces

In order to make clear the problem, we give firstly a prototype example. We have the identity by the division by zero calculus: For

$$f(z) = \frac{1+z}{1-z}, \quad f(1) = -1. \tag{13.1}$$

From the real part and imaginary part of the function, we have, for $z = x+iy$

$$\frac{1-x^2-y^2}{(1-x)^2+y^2} = -1, \quad \text{at} \quad (1,0) \tag{13.2}$$

and

$$\frac{y}{(1-x)^2+y^2} = 0, \quad \text{at} \quad (1,0), \tag{13.3}$$

respectively. Why the differences do happen?

In order to solve this problem, we will give the definition of the division by zero calculus on multidimensional spaces:

Definition of the division by zero calculus for multidimensional spaces. *For an analytic function $g(z)$ on a domain D on $\mathbf{C}^n, n \geq 1$, we set*

$$E = \{z \in D; g(z) = 0\}.$$

For an analytic function $f(z)$ on the set $D \setminus E$ such that

$$f(z) = \sum_{n=-\infty}^{\infty} C_n(z) g(z)^n, \tag{13.4}$$

for analytic functions $C_n(z)$ on D, we define the division by zero calculus by the correspondence:

$$f \longrightarrow F_{f,g=0}(z) := C_0(z)$$

that shows a natural analytic function of the function f on the domain D derived from $D \setminus E$ with respect to $E = \{z \in D; g(z) = 0\}$.

Of course, this definition is a natural extension of the one dimensional case. The expression (13.4) may be ensured by the general Laurent expansion that was introduced by Takeo Ohsawa:

Proposition 2. *In the Definition of the division by zero calculus for mutidimensional spaces, if the domain D is a regular domain, for any analytic function g, the expansion is possible.*

See [55] for the related topics.

However, since the uniqueness of the expansion is, in general, not valid, the division by zero calculus is not determined uniquely. However, we are very interested in the expansion (13.4) and the property of the function $C_0(z)$ as in the one dimensional case.

From the above arguments, we can see the desired results for the examples as follows:

$$\frac{1 - x^2 - y^2}{(1-x)^2 + y^2} \tag{13.5}$$

$$= -1 + \frac{2(1-x)}{(1-x)^2 + y^2} = -1, \quad \text{at} \quad (1,0)$$

and

$$\frac{y}{(1-x)^2 + y^2} = 0, \quad \text{at} \quad (1,0). \tag{13.6}$$

14 Division by zero calculus in physics

We will see the division by zero properties in various physical formulas. We found many and many division by zero in physics and others, however, we expect many publications by the related specialists. At the first stage, here we refer only to elementary formulas, as examples.

14.1 In balance of a steelyard

We will consider the balance of a steelyard and then we have the equation

$$aF_a = bF_b \tag{14.1}$$

as the moment equality, here, a, b are the distances from the fixed point and F_a, F_b are forces at the points a, b, respectively. Then, we have

$$F_a = \frac{b}{a}F_b. \tag{14.2}$$

For $a = 0$, should be considered as $F_a = 0$ by the division by zero $b/0 = 0$?

The identity (14.1) appears in many situations, and the above result may be valid similarly.

14.2 By rotation

We will give a simple physical model showing the result $\frac{0}{0} = 0$. We shall consider a disc with $x^2 + y^2 \leq a^2$ rowling uniformly with a positive constant angle velocity ω with the center at the origin. Then we see, at the only origin, $\omega = 0$ and at other all the points, ω is a constant. Then, we see, the velocity and the radius r are zero at the origin. This will mean that, in the general formula

$$v = r\omega,$$

or, in

$$\omega = \frac{v}{r}$$

at the origin,

$$\frac{0}{0} = 0.$$

We will not be able to obtain the result from

$$\lim_{r\to+0}\omega = \lim_{r\to+0}\frac{v}{r},$$

because it is the constant.

For a uniform rotation with velocity $\mathbf{v}$ with a center O' with a radius r. For the angular velocity vector ω and for the moving poisition P on the circle, we set $\mathbf{r} = OP$. Then,

$$\mathbf{v} = \omega \times \mathbf{r}. \tag{14.3}$$

If $\omega \times \mathbf{r} = 0$, then, of course, $\mathbf{v} = 0$.

14.3 By the Newton's Law

We will recall the fundamental law by Newton:

$$F = G\frac{m_1 m_2}{r^2} \tag{14.4}$$

for two masses m_1, m_2 with a distance r and a constant G. Of course,

$$\lim_{r\to+0} F = \infty, \tag{14.5}$$

however, as in our fraction

$$F = 0 = G\frac{m_1 m_2}{0}. \tag{14.6}$$

Of course, here, we can consider the above interpretation for the mathematical formula (14.4) as the new interpretation (14.6). In the ideal case, when the two masses are on the one point, the force F will not be positive, it will be reduced to zero.

In the Kepler (1571 - 1630) - Newton (1642 - 1727) law for central force movement of the planet,

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GmM}{r^3}\mathbf{r},$$

of course, we have $\mathbf{r} = \mathbf{0}$ for $r = 0$.

For the Coulomb's law, see similar formulas.

Indeed, in the formula

$$F = k\frac{(+q)(-q)}{r^2} \tag{14.7}$$

for $r = 0$, we have $F = 0$.

In general, in the formula

$$F = k\frac{(Q_1)(Q_2)}{r^2} \tag{14.8}$$

for $r = 0$, we have $F = 0$? (S. Senuma: 2016.8.20).

Furthermore, as well-known, the bright at a point at the distance r from the origin is given by the formula

$$B = k\frac{P}{r^2}, \tag{14.9}$$

where k is a constant and P is the amount of the light. Of course, we have, at the infinity:

$$B = 0. \tag{14.10}$$

Then, meanwhile, may we consider as

$$B = 0 \tag{14.11}$$

at the origin $r = 0$? Then we can obtain our formula

$$k\frac{P}{0} = 0,$$

as in our new formula.

14.4 An Interpretation of $0 \times 0 = 100$ from $100/0 = 0$

The expression $100/0 = 0$ will represent some divisor by the zero in a sense, not the usual one, and so, we will be able to consider some product sense $0 \times 0 = 100$.

We will show such interpretation.

We shall consider same two masses m, however, their constant velosities v for the origin are the same on the real line, in the symmetry way: We consider the mooving energy product E^2,

$$\frac{1}{2}mv^2 \times \frac{1}{2}m(-v)^2 = E^2. \tag{14.12}$$

We shall consider at the origin and we assume the two masses stop at the origin (possible in some case). Then, we can consider, formally

$$0 \times 0 = E^2. \tag{14.13}$$

The mooving energies turn to other energies, however, we can obtain some interpretation as in the above.

14.5 Capillary pressure in a narrow capillary tube

In a narrow capillary tube saturated with fluid such as water, the capillary pressure is simply expressed as follows,

$$Pc = \frac{2\sigma}{r} \tag{14.14}$$

where Pc is capillary pressure (suction pressure), σ is surface tension, and r is radius. If r is zero, there is no pressure. However Pc shows infinity, in the common meaning.

This simple equation is based on the Laplace-Young equation

$$P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{14.15}$$

where R_1 and R_2 are two principal radii of curvature at any point on the surface of a droplet or a bubble and in the case spherical form $R_1 = R_2 = R$. For a spherical bubble the pressure difference across the bubble film is zero as the pressure is the same on both sides of the film. The Laplace-Young equation reduces to

$$\frac{1}{R_1} + \frac{1}{R_2} = 0. \tag{14.16}$$

On other hand when diameter of a bubble is decreased and becomes $0 (R = 0)$, the bubbles collapse, enormous energy is generated. Accumulated free energy in the bubble is released instantaneously.

14.6 Circles and curvature - an interpretation of the division by zero $r/0 = 0$

We consider a solid body called right circular cone whose bottom is a disc with radius r_2. We cut the body with a disc of radius $r_1 (0 < r_1 < r_2)$ that is parallel to the bottom disc. We denote the distance by d between the both discs and R the distance between the top point of the cone and the bottom circle on the surface of the cone. Then, R is calculated by Eko Michiwachi (8 years old daughter of Mr. H. Michiwaki) as follows:

$$R = \frac{r_2}{r_2 - r_1}\sqrt{d^2 + (r_2 - r_1)^2},$$

that is called *EM radius*, because by the rotation of the cone on the plane, the bottom circle writes the circle of radius R. We denote by $K = K(R) = 1/R$ the curvature of the circle with radius R. We fix the distance d. Now note that:

$$r_1 \to r_2 \Longrightarrow R \to \infty.$$

This will be natural in the sense that when $r_1 = r_2$, the circle with radius R becomes a line.

However, the division by zero will mean that when $r_1 = r_2$, the above EM radius formula makes sense and $R = 0$. What does it mean? Here, note that, however, then the curvature $K = K(0) = 0$ by the division by zero; that is, the circle with radius R becomes a line, similarly. The curvature of a point (circle of radius zero) is zero.

14.7 Vibration

In the typical ordinary differential equation

$$m\frac{d^2x}{dt^2} = -kx, \tag{14.17}$$

we have a general solution

$$x = C_1 \cos(\omega t + C_2), \quad \omega = \sqrt{\frac{k}{m}}. \tag{14.18}$$

If $k = 0$, that is, if $\omega = 0$, then the period T is given by

$$T = \frac{2\pi}{\omega}.$$

Then, should be understood as $T = 0$, no period?

In the typical ordinary differential equation

$$m\frac{d^2x}{dt^2} + kx = f \cos \omega t, \tag{14.19}$$

we have a special solution

$$x = \frac{f}{m} \frac{1}{|\omega^2 - \omega_0^2|} \cos \omega t, \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{14.20}$$

Then, how will be the case

$$\omega = \omega_0 \tag{14.21}$$

?

For example, for the differential equation

$$y'' + a^2 y = b \cos \lambda x, \tag{14.22}$$

we have a special solution, with the condition $\lambda \neq a$

$$y = \frac{b}{a^2 - \lambda^2} \cos \lambda x. \tag{14.23}$$

Then, when $\lambda = a$, by the division by zero, we obtain the special solution

$$y = \frac{bx \sin(ax)}{2a} + \frac{b \cos ax}{4a^2}. \tag{14.24}$$

14.8 Spring or circut

We will consider a spring with two spring constants $\{k_j\}$ in a line. Then, the spring constant k of the spring is given by the formula

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}, \tag{14.25}$$

by Hooke's law. We know, in particular, if $k_1 = 0$, then

$$\frac{1}{k} = \frac{1}{0} + \frac{1}{k_2}, \tag{14.26}$$

and by the division by zero,

$$k = k_2, \tag{14.27}$$

that is very reasonable. In particular, by Hooke's law, we see that

$$\frac{0}{0} = 0. \tag{14.28}$$

The corresponding result for the case of Ohmu's law is similar and valid.

14.9 Motion

A and B start at the origin on the real positive axis with, for $t = 0$

$$\frac{d^2x}{dt^2} = a, \quad \frac{dx}{dt} = u$$

and

$$\frac{d^2x}{dt^2} = b, \quad \frac{dx}{dt} = v,$$

respectively. After the time T and at the distance X from the origin, if they meet, then we obtain the relations

$$T = \frac{2(u-v)}{b-a}$$

and

$$X = \frac{2(u-v)(ub-va)}{(b-a)^2}.$$

For the case $a = b$, we obtain the reasonable solutions $T = 0$ and $X = 0$.

We will consider the motion (x, y) represented by $x = \cos\theta, y = \sin\theta$ from $(1, 0)$ to $(-1, 0)$ $(0 \leq \theta \leq \pi)$ with the condition

$$v_x = \frac{dx}{dt} = -\sin\theta\frac{d\theta}{dt} = V \quad \text{(constant)}. \tag{14.29}$$

Then, we have:

$$v_y = \frac{dy}{dt} = -V\frac{1}{\tan\theta}, \tag{14.30}$$

and

$$a_y = \frac{d^2y}{dt^2} = -V^2\frac{1}{\sin^3\theta}. \tag{14.31}$$

Then we see that:

$$v_y(1,0) = 0, \text{that is, } \frac{1}{\tan 0} = 0, \tag{14.32}$$

$$v_y(-1,0) = 0, \text{that is, } \frac{1}{\tan\pi} = 0, \tag{14.33}$$

$$a_y(1,0) = 0, \text{that is, } \frac{1}{\sin^3 0} = 0, \tag{14.34}$$

and

$$a_y(-1,0) = 0, \text{that is, } \frac{1}{\sin^3\pi} = 0. \tag{14.35}$$

Here, we interpretated, by the physical reason, that

$$\frac{1}{\sin^3\theta} = \frac{1}{\sin\theta}\cdot\frac{1}{\sin\theta}\cdot\frac{1}{\sin\theta}. \tag{14.36}$$

If we consider

$$\left[\frac{1}{\sin^3\theta}\right], \tag{14.37}$$

by means of the Laurent expansion, then we have another value. For example,

$$\frac{1}{\sin^2 z} = \sum_{\nu=-\infty}^{\infty}\frac{1}{(z-\nu\pi)^2}, \tag{14.38}$$

$$\left(\frac{1}{\sin^2 z}\right)(0) = \frac{2}{\pi^2}\sum_{\nu=1}^{\infty}\frac{1}{\nu^2} = \frac{1}{3}.$$

We can find many and many the division by zero and division by zero calculus in physics.

15 Interesting examples in the division by zero

We will give interesting examples in the division by zero. Indeed, the division by zero may be looked in the elementary mathematics and also in the universe.

- For the line
$$\frac{x}{a} + \frac{y}{b} = 1, \tag{15.1}$$
if $a = 0$, then by the division by zero, we have the line $y = b$. This is a very interesting property creating new phenomena at the term x/a for $a = 0$.

  Note that here we can not consider the case $a = b = 0$.

- For the area $S(a, b) = ab$ of the rectangle with sides of lengths a, b, we have
$$a = \frac{S(a, b)}{b} \tag{15.2}$$
and for $b = 0$, formally
$$a = \frac{0}{0}. \tag{15.3}$$
However, there exists a contradiction. $S(a, b)$ depends on b and by the division by zero calculus, we have, for the case $b = 0$, the right result
$$\frac{S(a, b)}{b} = a. \tag{15.4}$$

- For the identity
$$(a^2 + b^2)(a^2 - b^2) = c^2(a^2 - b^2); a, b, c > 0 \tag{15.5}$$
if $a \neq b$, then we have the Pythagorean theorem
$$a^2 + b^2 = c^2. \tag{15.6}$$
However, for the case $a = b$, we have also the Pythagorean theorem, by the division by zero calculus
$$2a^2 = c^2. \tag{15.7}$$

- Let $\alpha_j; j = 1, ..., n$ be the solutions of the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... a_0 = 0, \quad a_n \neq 0, \tag{15.8}$$

then, $\frac{1}{\alpha_j}; j = 1, ..., n$ are the solution of the equation

$$f\left(\frac{1}{x}\right) = 0, \tag{15.9}$$

when we apply the division by zero.

- In a Hilbert space H, for a fixed member v and for a given number d we set

$$V = \{y \in H; (y, v) = d\} \tag{15.10}$$

and for fixed $x \in H$

$$d(x, V) := \frac{|(x, v) - d|}{\|v\|}. \tag{15.11}$$

If $v = 0$, then, $(y, v) = 0$ and d have to zero. Then, since $H = V$, we have

$$0 = \frac{0}{0}. \tag{15.12}$$

- For the equation

$$\mathbf{a} \times \mathbf{x} = \mathbf{b}$$

the solutions exist if and only if $\mathbf{a} \cdot \mathbf{b} = 0$ and then, we have

$$\mathbf{x} = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}.$$

For $\mathbf{a} = \mathbf{0}$, we have $\mathbf{x} = \mathbf{0}$ by the division by zero.

- We consider 4 lines

$$\begin{aligned} a_1 x + b_1 y + c_1 = 0, \\ a_1 x + b_1 y + c_1' = 0, \\ a_2 x + b_2 y + c_2 = 0, \\ a_2 x + b_2 y + c_2' = 0, \end{aligned} \tag{15.13}$$

Then, the area S surrounded by these lines is given by the formula

$$S = \frac{|c_1 - c_1'| \cdot |c_1 - c_1'|}{|a_1 b_2 - a_2 b_1|}. \tag{15.14}$$

Of course, if $|a_1 b_2 - a_2 b_1| = 0$, then $S = 0$.

- $\frac{1}{\sin 0} = \frac{1}{\cos \pi/2} = 0$. Consider the linear equation with a fixed positive constant a

$$\frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1. \tag{15.15}$$

Then, the results are clear from the graphic meanings.

- For the tangential line at a point $(a\cos\theta, \sin\theta)$ on the elliptic curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0 \tag{15.16}$$

we have $Q(a/(\cos\theta), 0)$ and $R(0, b/(\sin\theta))$ as the common points with x and y axisis, respectively. if $\theta = 0$, then $Q(a, 0)$ and $R(0, 0)$. If $\theta = \pi/2$, then $Q(0, 0)$ and $R(0, b)$.

- For the tangential line at the point $(a\cos\theta, \sin\theta)$ on the elliptic curve, we shall consider the area $S(\theta)$ of the triangle formed by this line and x, y axises

$$S(\theta) = \frac{ab}{|\sin\theta|}.$$

Then, by the division by zero calculus, we have $S(0) = 0$.

- The common point of B (resp. B') of a tangential line (15.16) and the line $x = a$ (resp. $x = -a$) is given by

$$B\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right).$$

(resp.

$$B'\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right).$$

) The circle with diameter BB' is given by

$$x^2 + y^2 - \frac{2b}{\sin\theta} y - (a^2 - b^2) = 0.$$

Note that this circle passes two forcus points of the elliptic curve. Note that for $\theta = 0$, we have the reasonable result, by the division by zero calculus

$$x^2 + y^2 - (a^2 - b^2) = 0.$$

In the classical theory for quadratic curves, we have to arrange globally it by the division by zero calculus.

- The area $S(x)$ surrounded by two x, y axises and the line passing a fixed point $(a, b), a, b > 0$ and a point $(x, 0)$ is given by

$$S(x) = \frac{bx^2}{2(x-a)}. \tag{15.17}$$

For $x = a$, we obtain, by the division by zero calculus, the very interesting value

$$S(a) = ab. \tag{15.18}$$

- For example, for fixed point $(a, b); a, b > 0$ and fixed a line $y = (\tan\theta)x, 0 < \theta < \pi$, we will consider the line $L(x)$ passing the two points (a, b) and $(x, 0)$. Then, the area $S(x)$ of the triangle surround by the three lines $y = (\tan\theta)x$, $L(x)$ and the x axis is given by

$$S(x) = \frac{b}{2}\frac{x^2}{x - (a - b\cot\theta)}.$$

For the case $x = a - b\cot\theta$, by the division by zero calculus, we have

$$S(a - b\cot\theta) = b(a - b\cot\theta).$$

Note that this is the area of the parallelogram through the origin and the point (a, b) formed by the lines $y = (\tan\theta)x$ and the x axis.

- We consider the regular triangle with the vertexes $(-a/2, \sqrt{3}a/2)$, $(a/2, \sqrt{3}a/2)$. Then, the area $S(h)$ of the triangle surrounded by the three lines that the line through $(0, h+\sqrt{3}a/2)$ and $(-a/2, \sqrt{3}a/2)$, the line through $(0, h+\sqrt{3}a/2)$ and $(a/2, \sqrt{3}a/2)$ and the x- axis is given by

$$S(h) = \frac{\left(h + (\sqrt{3}/2)a\right)^2}{2h}. \tag{15.19}$$

Then, by the division by zero calculus, we have, for $h = 0$,

$$S(0) = \frac{\sqrt{3}}{2}a^2.$$

- Similarly, we will consider the cone formed by the rotation of the line

$$\frac{kx}{a(k+h)} + \frac{y}{k+h} = 1$$

and the x, y plane with center the z- axis ($a, h > 0$, and a, h are fixed). Then, the volume $V(x)$ is given by

$$V(k) = \frac{\pi}{3}\frac{a^2(k+h)^3}{k^2}.$$

Then, by the division by calculus, we have the reasonable value

$$V(0) = \pi a^2 h.$$

- In the formula

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, \tag{15.20}$$

for $\theta = \pi$, we have: 0=0/0.

- In the formula

$$\tan z_1 \pm \tan z_2 = \frac{\sin(z_1 + z_2)}{\sin z_1 \sin z_2}, \tag{15.21}$$

for $z_1 = \pi/2, z_2 = 0$, we have: 0=1/0.

- In the formula

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}, \tag{15.22}$$

for $\theta = \pi/4$, we have: 0=2/0.

- In the elementary identity

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}, \tag{15.23}$$

for the case $\alpha = \beta = \pi/2$, we have

$$\tan\frac{\pi}{2} = \frac{1+1}{1-1\cdot 1} = \frac{2}{0} - 0. \tag{15.24}$$

- For the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n, \tag{15.25}$$

we have, of course, $\lim_{n\to\infty} a_n = e$. Meanwhile, by formally, we have

$$a_0 = \left(1 + \frac{1}{0}\right)^0 = 1^0 = 1.$$

However, we obtain

$$a_0 = \exp\left\{n \log\left(1 + \frac{1}{n}\right)\right\}_{n=0} = e, \tag{15.26}$$

by the division by zero calculus. Indeed, for $x = 1/n$, we have

$$n \log\left(1 + \frac{1}{n}\right) = \frac{1}{x}\left(x - \frac{x^2}{2} + ...\right)$$

and this equals 1 for the point at infinity, by the division by zero calculus. Note that for the definition by exponential functions by (15.26) is fundamental.

- For example, for the plane equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \tag{15.27}$$

for $a = 0$, we can consider the line naturally, by the division by zero

$$\frac{y}{b} + \frac{z}{c} = 1. \tag{15.28}$$

- For the Gauss map function

$$f(x) = \frac{1}{x} - \left[\frac{1}{x}\right], \tag{15.29}$$

we have, automatically, by the division by zero

$$f(0) = 0.$$

- For the product and sum representations

$$\Pi_{\nu=-\infty,\nu\neq 0}^{\infty}\left(1-\frac{z}{\nu\pi}\right)\exp\frac{z}{\nu\pi} \tag{15.30}$$

and

$$\sum_{\nu=-\infty,\nu\neq 0}^{\infty}\left(\log\left(1-\frac{z}{\nu\pi}\right)+\frac{z}{\nu\pi}\right), \tag{15.31}$$

we do not need the conditions $\nu \neq 0$, because, the corresponding terms are automatically 1 and zero, respectively, by the division by zero.

- For the solutions

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{15.32}$$

of the quadratic equation

$$ax^2+bx+c=0, \tag{15.33}$$

we have, the solution, for $a=0$ and $b\neq 0$,

$$x=-\frac{c}{b},$$

by the division by zero calculus.

- Let X be a nonnegative random variable with a continuous distribution F, then the mean residual life function $M(x)$ is given by, if $1-F(x)>0$,

$$M(x)=\frac{\int_x^{\infty}(1-F(\xi))d\xi}{1-F(x)}. \tag{15.34}$$

However, if $1-F(x)=0$, automatically, we have $M(x)=0$, by the division by zero.

- As in the line case, in the hyperbolic curve

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad a,b>0, \tag{15.35}$$

by the representations by parameters

$$x=\frac{a}{\cos\theta}=\frac{a}{2}\left(\frac{1}{t}+t\right)$$

and

$$y = \frac{b}{\tan\theta} = \frac{b}{2}\left(\frac{1}{t} - t\right)$$

the origin $(0, 0)$ may be included as the point of the hyperbolic curve, as we see from the cases $\theta = \pi/2 = 0$ and $t = 0$.

In addition, from the fact, we will be able to understand that the asymptotic lines are the tangential lines of the hyperbolic curve.

The two tangential lines of (15.35) with gradient m is given by

$$y = mx \pm \sqrt{a^2m^2 - b^2} \tag{15.36}$$

and the gradients of the asymptotic lines are

$$m = \pm\frac{b}{a}. \tag{15.37}$$

Then, we have asymptotic lines $y = \pm\frac{b}{a}x$ as tangential lines in (14.35).

The common points of (15.35) and (15.36) are given by

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \pm\frac{b^2m}{\sqrt{a^2m^2 - b^2}}\right). \tag{15.38}$$

For the case $a^2m^2 - b^2 = 0$, we have they are $(0, 0)$.

- We fix a circle

$$x^2 + (y - a)^2 = a^2, \quad a > 0. \tag{15.39}$$

At the point $(2a + d, 0), d > 0$, we consider two tangential lines for the circle. Let 2θ is the angle between two tangential lines at the point $(2a+d, 0)$, Then, the area $S(h) = S(\theta)$ and the length $L(x) = L(\theta)$ are given by

$$S(h) = S(\theta) = \frac{a}{\sqrt{h}}(h + 2a)^{\frac{3}{2}} \tag{15.40}$$

$$= \frac{a^2}{\cos\theta}\left(\sin\theta + 2 + \frac{1}{\sin\theta}\right)$$

and

$$L(h) = L(\theta) = \frac{a}{\sqrt{h}}\sqrt{h + 2a} \tag{15.41}$$

$$= a\left(\frac{1}{\cos\theta} + \tan\theta\right),$$

respectively. For $h = 0$ and $\theta = 0$, by diivision by zero calculus, we see that all are zero.

- We consider two spheres defined by

$$x^2 + y^2 + z^2 + 2a_j + 2b_j + 2c_j + 2d_j = 0, \quad j = 1, 2. \tag{15.42}$$

Then, the angle θ by two spheres is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2 - (d_1 + d_2)}{\sqrt{a_1^2 + b_1^2 + c_1^2 - 2d_1}\sqrt{a_2^2 + b_2^2 + c_2^2 - 2d_2}}. \tag{15.43}$$

If $\cos\theta = 0$, then, two spheres are orthogonal or one sphere is a point sphere.

- For the parabolic equation

$$y^2 = 4px,$$

two points $(pt^2, 2pt)$ and $(qt^2, 2qt)$ is a diameter is if and only if

$$(s - t)\{t(s + t) + 2\} = 0; \quad s = -t - \frac{2}{t}$$

and the diameter r is given by

$$r^2 = p^2(t - s)^2\{(t + s)^2 + 4\}.$$

Here, we should consider the case $t = s = 0$ as $r = 0$ and

$$0 = -0 - \frac{2}{0},$$

and the x and y axises are the orthogonal two tangential lines of the parabolic equation.

- For the integral equation, for a constant k

$$\int_0^x y dx = ky,$$

we have the general solution

$$y = C \exp \frac{x}{k}.$$

If $k = 0$, then, of course, we have, $y = C$.

For the integral equation

$$\int_0^x y dx = k \int_0^x \sqrt{1 + (y')^2} dx$$

we have the solution

$$y = \frac{k}{2} \left(\exp \frac{x}{k} + \exp - \frac{x}{k} \right).$$

If $k = 0$, then, we should have $y = 0$.

16 What is the zero?

The zero 0 as the complex number or real number is given clearly by the axioms by the complex number field and real number field.

For this fundamental idea, we should consider the **Yamada field** containing the division by zero. The Yamada field and the division by zero calculus will arrange our mathematics, beautifully and completely; this will be our real and complete mathematics.

Standard value

The zero is a center and stand point (or bases, a standard value) of the coordinates - here we will consider our situation on the complex or real 2 dimensional spaces. By stereographic projection mapping or the Yamada field, the point at infinity 1/0 is represented by zero. The origin of the coordinates and the point at infinity correspond each other.

As the standard value, for the point $\omega_n = \exp\left(\frac{\pi}{n}i\right)$ on the unit circle $|z| = 1$ is for $n = 0$:

$$\omega_0 = \exp\left(\frac{\pi}{0}i\right) = 1, \quad \frac{\pi}{0} = 0. \tag{16.1}$$

For the mean value

$$M_n = \frac{x_1 + x_2 + ... + x_n}{n},$$

we have

$$M_0 = 0 = \frac{0}{0}.$$

Fruitful world

For example, very and very general partial differential equations, if the coefficients or terms are zero, we have some simple differential equations and the extreme case is all the terms are zero; that is, we have trivial equations $0 = 0$; then its solution is zero. When we see the converse, we see that the zero world is a fruitful one and it means some vanishing world. Recall Yamane phenomena, the vanishing result is very simple zero, however, it is the result from some fruitful world. Sometimes, zero means void or nothing world, however, it will show some changes as in the Yamane phenomena.

From 0 to 0; 0 means all and all are 0

As we see from our life figure, a story starts from the zero and ends to the zero. This will mean that 0 means all and all are 0. The zero is a mother of all.

Impossibility

As the solution of the simplest equation

$$ax = b \tag{16.2}$$

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (16.2) is impossible. We saw for different parallel lines or different parallel planes, their common points are the origin. Certainly they have the common points of the point at infinity and the point infinity is represented by zero. However, we can understand also that they have no solutions, no common points, because the point at infinity is an ideal point.

We will consider the simple differential equation

$$m\frac{d^2x}{dt^2} = 0, m\frac{d^2y}{dt^2} = -mg \tag{16.3}$$

with the initial conditions, at $t = 0$

$$\frac{dx}{dt} = v_0 \cos\alpha, \frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = 0. \tag{16.4}$$

Then, the highest high h, arriving time t, the distance d from the starting point origin to the point $y(2t) = 0$ are given by

$$h = \frac{v_0 \sin^2\alpha}{2g}, d = \frac{v_0 \sin\alpha}{g} \tag{16.5}$$

and

$$t = \frac{v_0 \sin\alpha}{g}. \tag{16.6}$$

For the case $g = 0$, we have $h = d = t = 0$. We considered the case that they are the infinity; however, our mathematics means zero, which shows impossibility.

These phenomena were looked many cases on the universe; it seems that God does not like the infinity.

17 Conclusion

Apparently, the common sense on the division by zero with a long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on derivatives we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero.

This book is an elementary mathematics on our division by zero as the first publication of books for the topics. The contents have wide connections to various fields beyond mathematics. The author expects the readers write some philosophy, papers and essays on the division by zero from this simple source book.

The division by zero theory may be developed and expanded greatly as in the author's conjecture whose break theory was recently given surprisingly and deeply by Professor Qi'an Guan [18] since 30 years proposed in [47] (the original is in [46]).

We have to arrange globally our modern mathematics with our division by zero in our undergraduate level.

We have to change our basic ideas for our space and world.

We have to change globally our textbooks and scientific books on the division by zero.

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